Exclusive or inclusive? The lattice might have the answer...

Cluster of Excellence PRISMA+ Precision Physics, Fundamental Interactions and Structure of Matter

JOHANNES GUTENBERG UNIVERSITÄT MAINZ

JG|U

21.05.2025

Andreas Jüttner

CERN

University of Southampton

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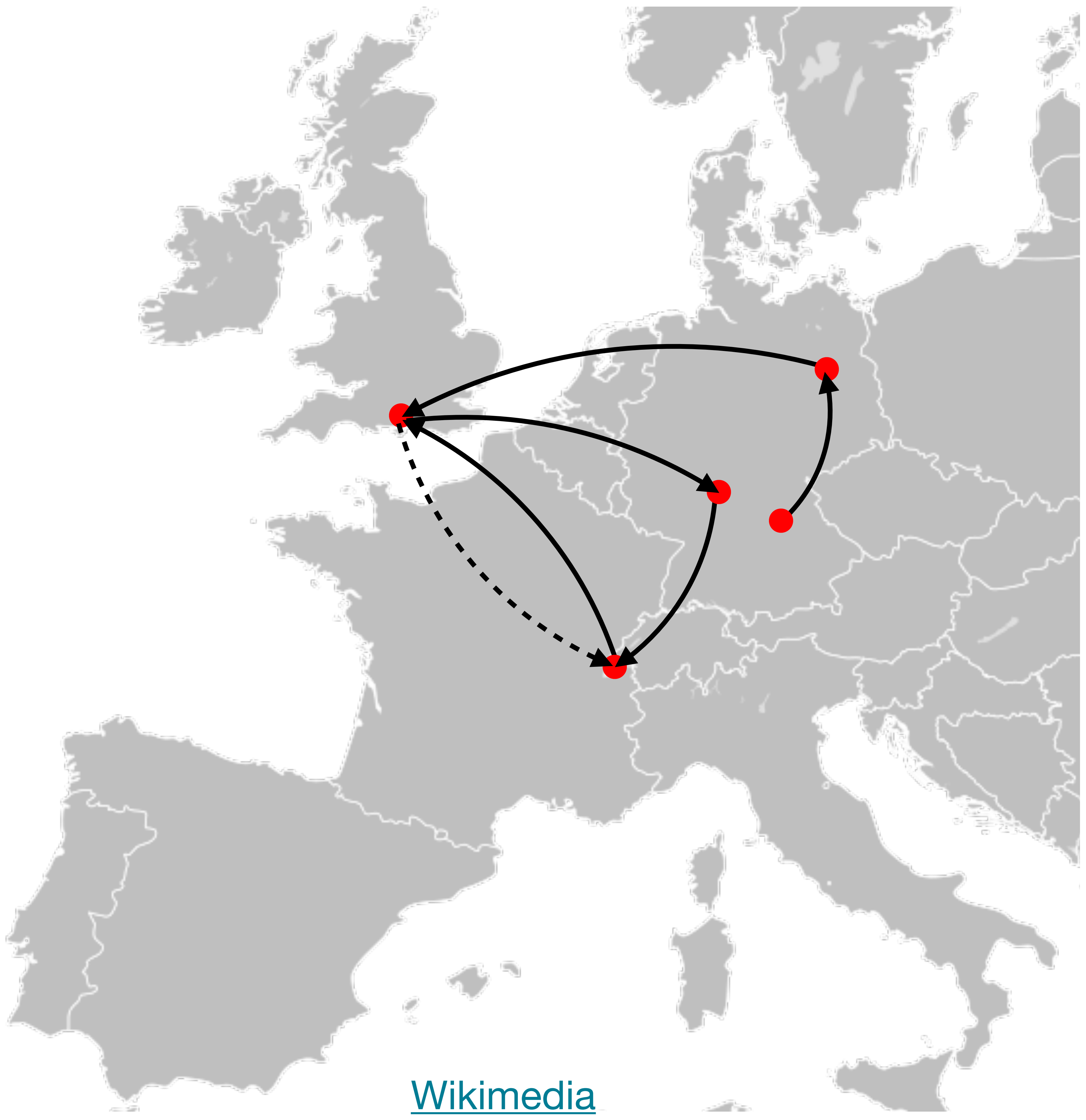
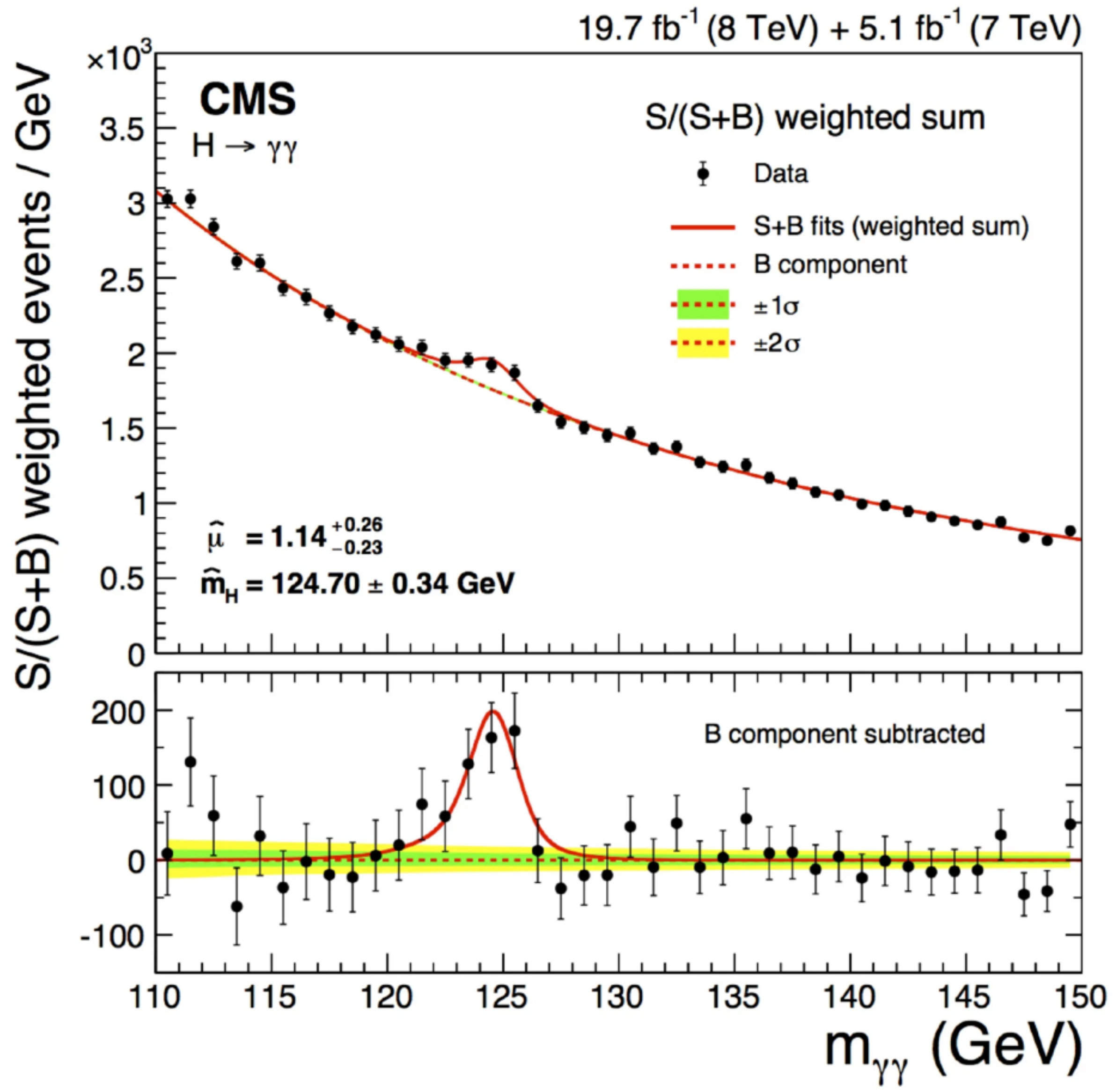
Career path

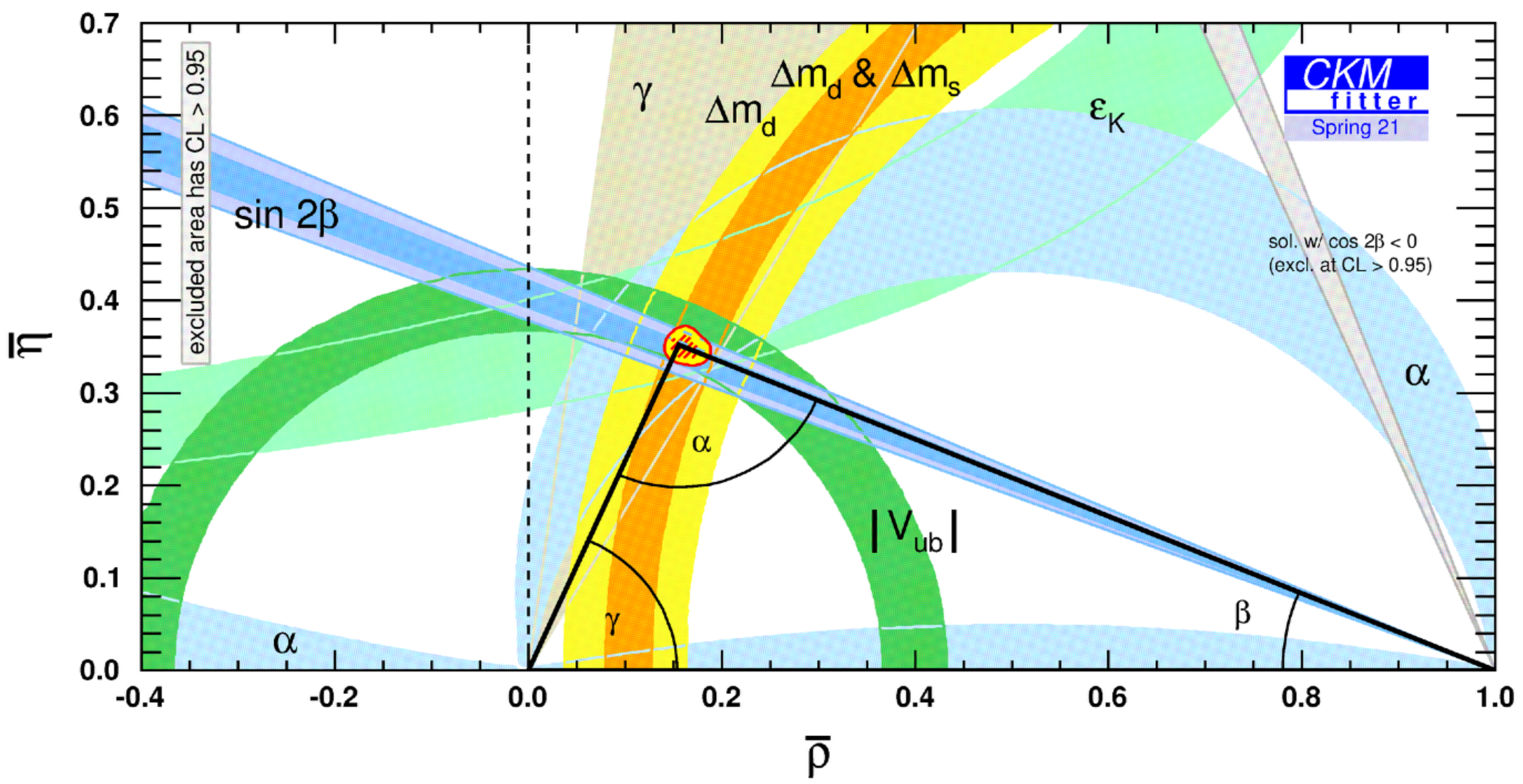


- Erlangen-Nürnberg: undergraduate studies
- Humboldt Berlin: PhD (2002-2004)
- **postdocs**:
  Southampton (2004-2007)
  Mainz (2007-2009)
  CERN Fellow (2009-2012)
- **staff**:
  Southampton (ERC Starting Grant 2011)
  since 2021: on leave from personal chair in Southampton
  currently CERN Staff (Limited Duration)

Testing the Standard Model

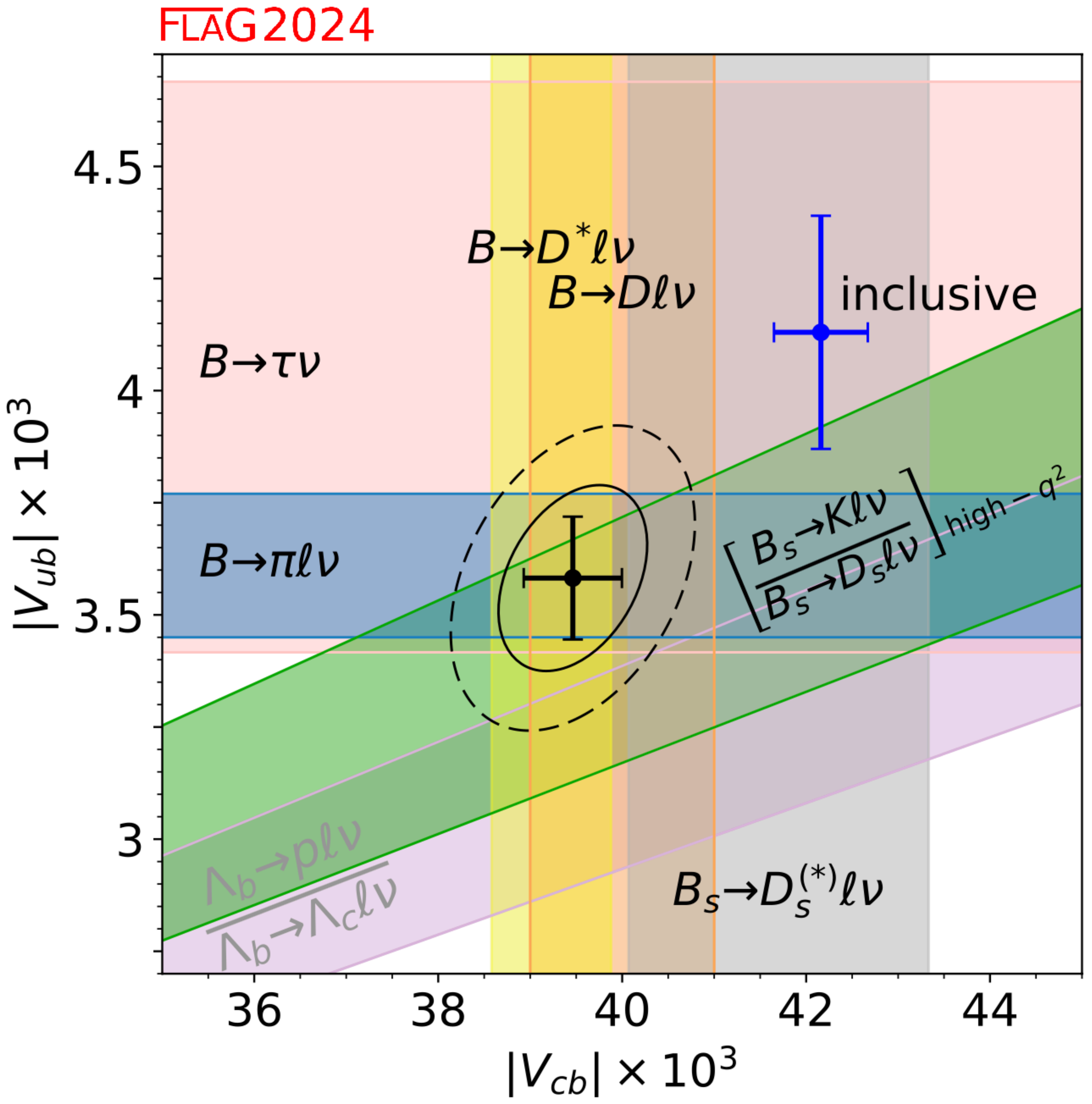
- searches for new physics
  - *direct searches* — ’bump in the spectrum’
  - *indirect searches* — SM provides relations between processes; we can therefore use experiment + theory to over-constrain SM





We will now look at Lattice QCD’s role in *indirect searches*

Testing the Standard Model



In this talk — various ways to look at the $|V_{cb}|$ tension

A) Exclusive decay $B \to D^* \ell \bar{\nu}_\ell$:

- new quality of experimental data
- new quality of lattice data

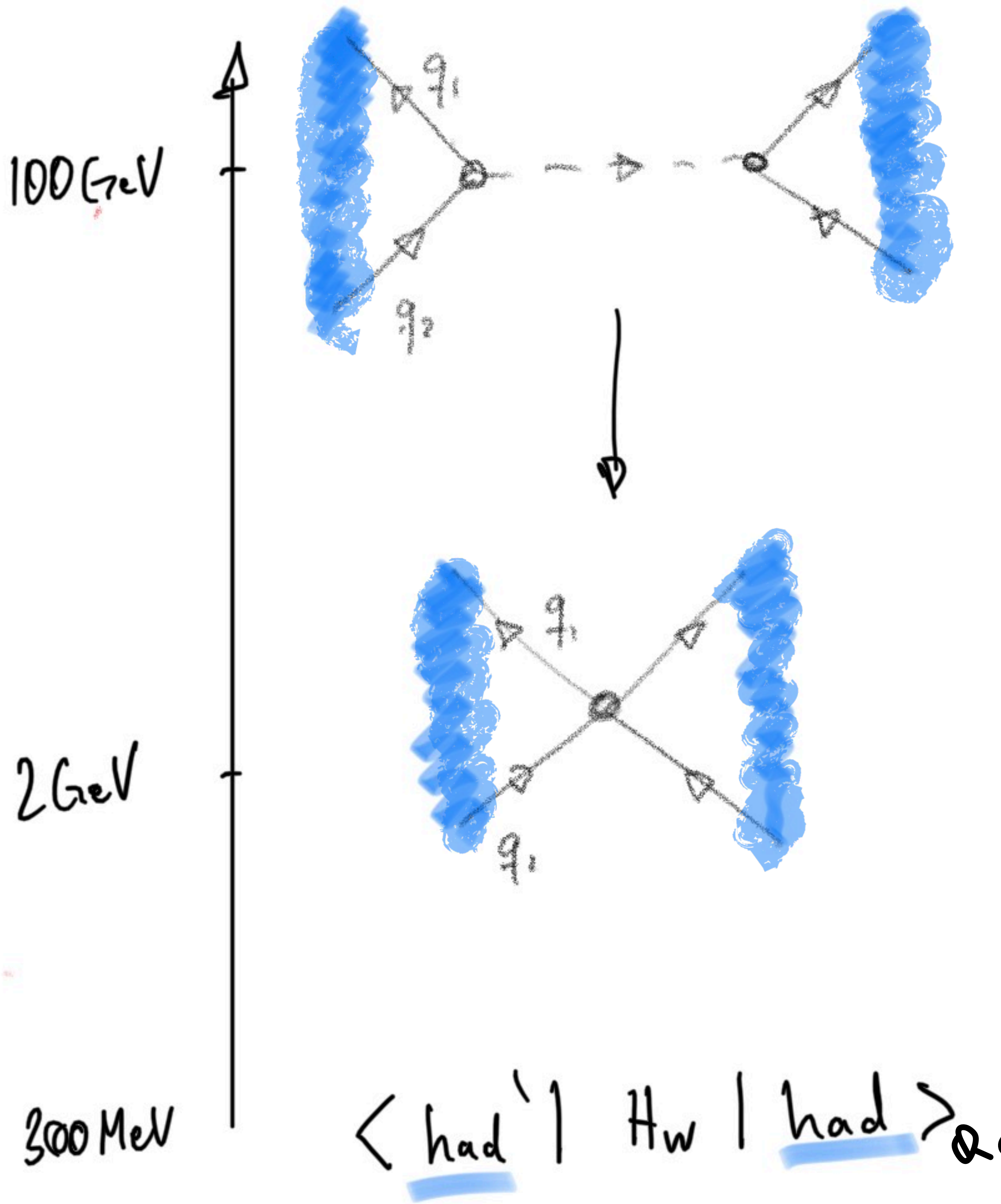
→ discuss new and improved analysis techniques

B) Inclusive decay $B \to X_c \ell \bar{\nu}_\ell$:

- existing determinations OPE based
- new ideas allow for lattice computations

→ discuss new ideas and preliminary results

SM theory



$$\langle \text{had}' | H_W | \text{had} \rangle_{QCD+QED}$$

$$\langle 0 | H_W | \text{had} \rangle_{QCD+QED}$$

$$\langle \text{had}' | T[H_W H_W'] | \text{had} \rangle_{QCD+QED}$$

Theory can be hard:

- all three sectors of SM contribute
- accuracy important

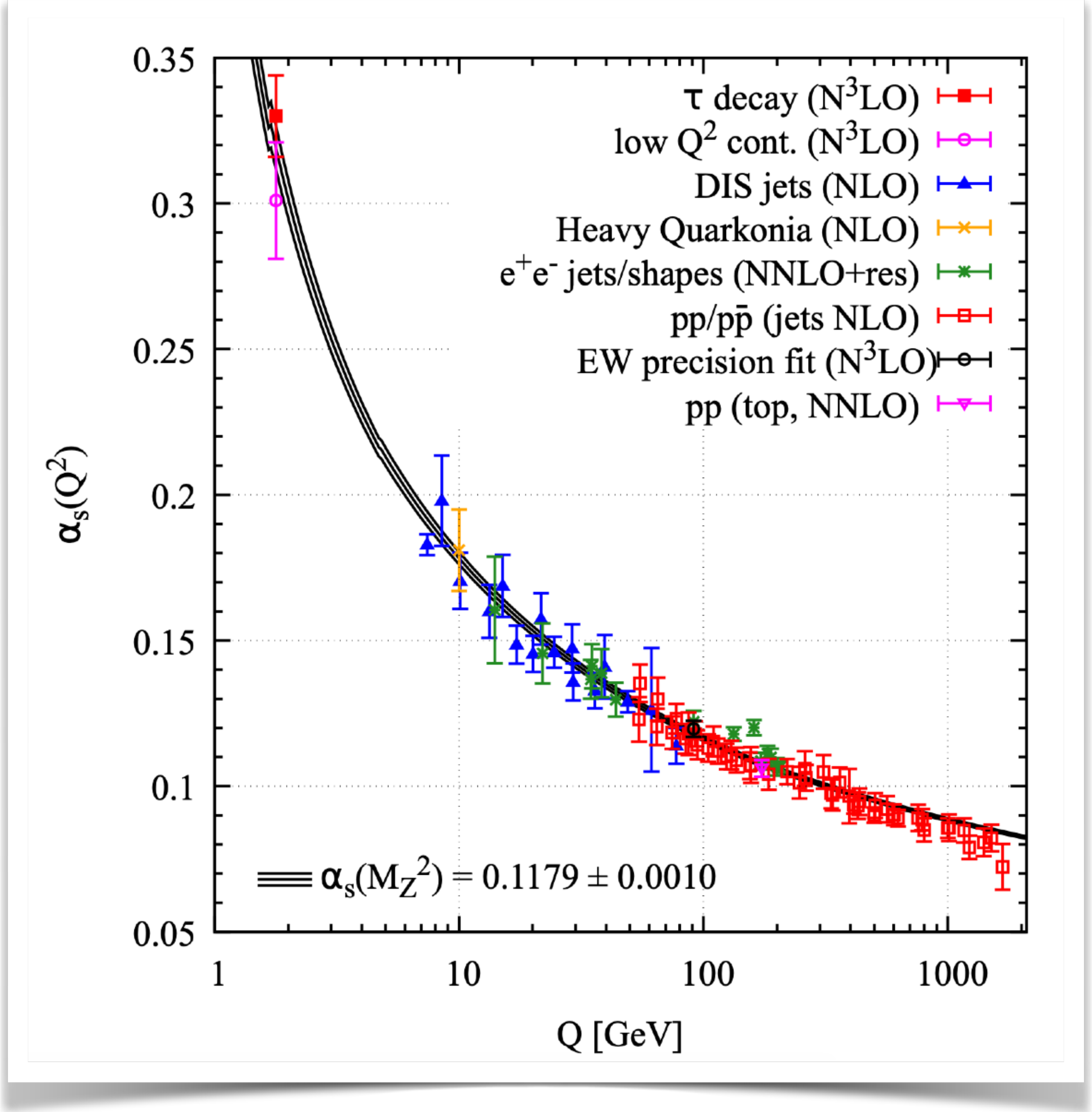
But SM helps us a bit:

- Weak gauge bosons so heavy that we can replace them by point-interaction described by an Effective Hamiltonian H_W (conveniently we thereby *get rid* of a very high energy scale)

Theory predictions require computations in weak eff. theory, QCD and QED

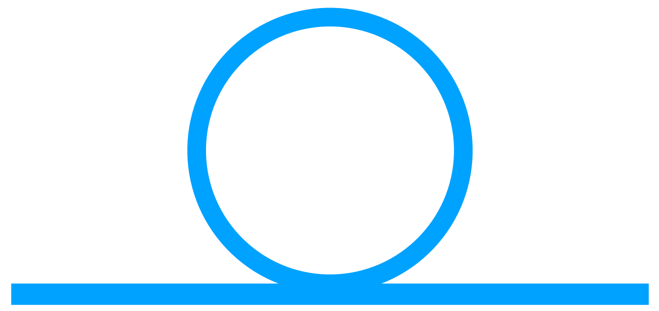
SM sectors

SM-sector	typical coupling	mediator
WEAK	10^{-5}GeV^{-2}	Z, W$^\pm$
EM	1/137	γ
QCD	**0-*O(1)***	gluons

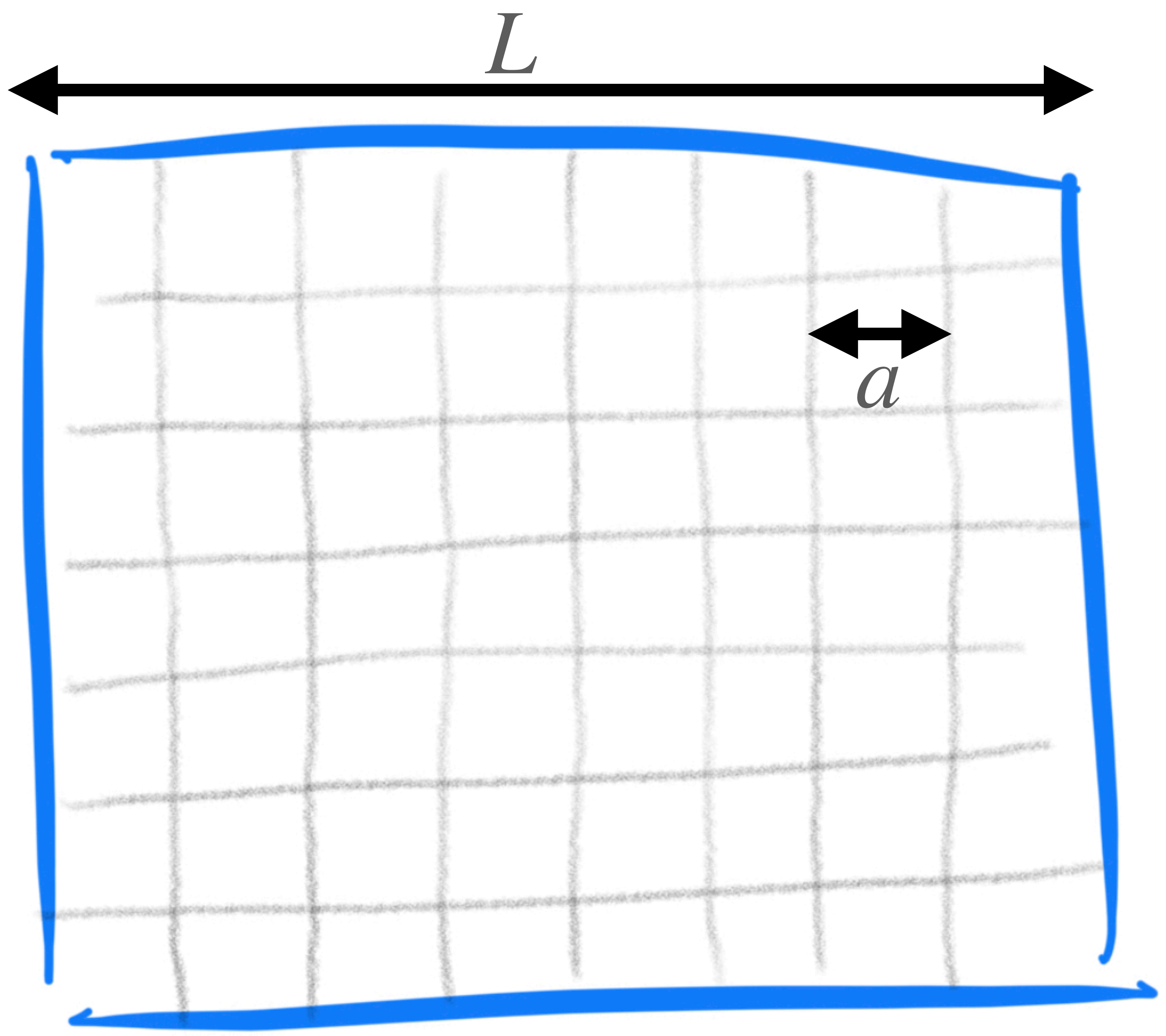


hadronic uncertainties require nonperturbative methods → Lattice QCD

A regularisation…



dimensional regularisation, cutoff, mass-term, …, or



$$\frac{1}{L} \ll \Lambda_{\text{physics}} \ll \frac{1}{a}$$

$$\int dx\, f(x) \to a \sum_x f(x) \qquad \partial_\mu f(x) \to \frac{f(x+a) - f(x)}{a}$$

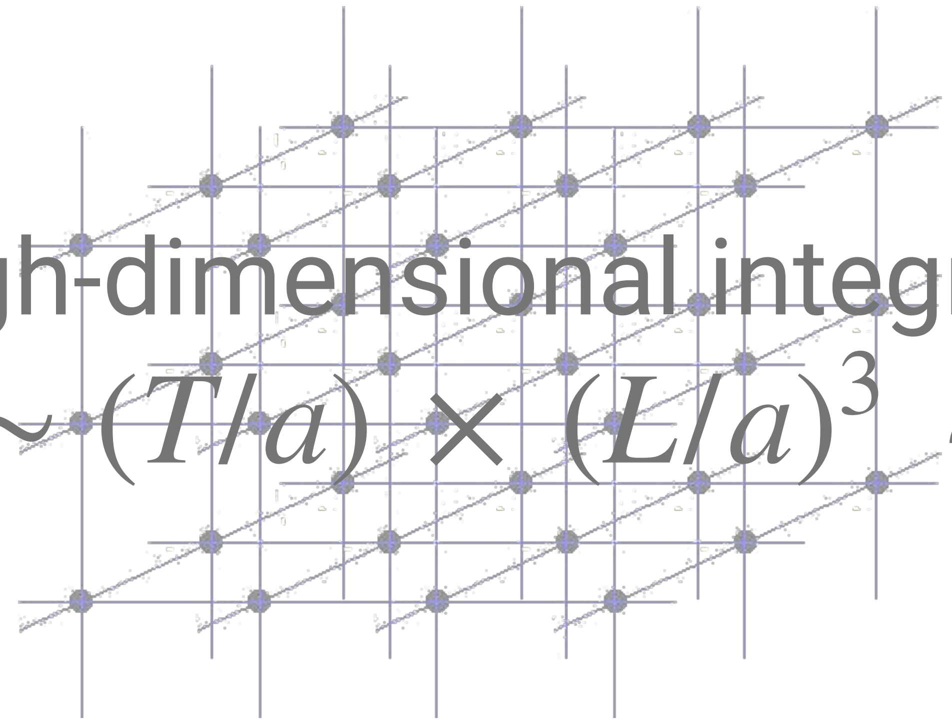
What to do with it?

Allows path integral quantisation $\langle 0|O|0\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}[U,\psi,\bar{\psi}]\, O\, e^{\,iS_{\text{lat}}[U,\psi,\bar{\psi}]}$

- can do perturbation theory

- can do nonPT calculations $\langle 0|O|0\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}[U,\psi,\bar{\psi}]\, O\, \boxed{e^{-S_{\text{lat}}[U,\psi,\bar{\psi}]}}$

Euclidean space-time Boltzmann factor



high-dimensional integral:

$$N_{\text{sim}} \sim (T/a) \times (L/a)^3 \sim 10^9$$

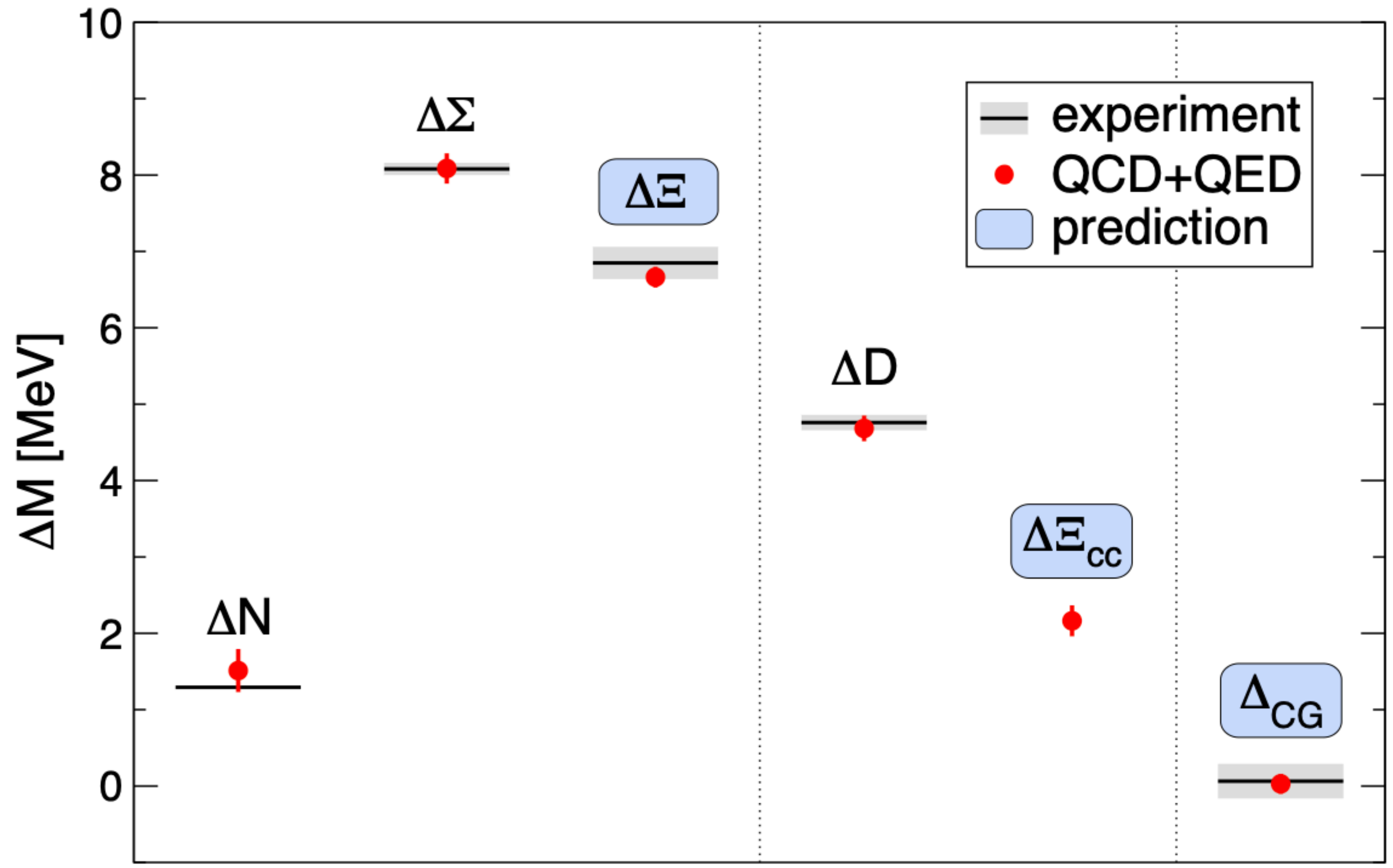
Lattice QCD

$$\langle 0 | O | 0 \rangle = \frac{1}{\mathcal{Z}} \int D[A, \bar{\psi}, \psi] \, O \, e^{-S_{\mathrm{QCD}}[A,\bar{\psi},\psi]}$$

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + \sum_f \bar{\psi}_f \left(i\gamma^\mu D_\mu - m_f \right) \psi_f$$

Free parameters:

- gauge coupling $g \rightarrow \alpha_s = g^2/4\pi$
- quark masses $m_f = u,d,s,c,b,t$



BMW Collaboration Science 347 (2015) 1452-1455

- Lagrangian of massless gluons and *almost massless quarks*
- What experiment sees are bound states, e.g. $m_\pi, m_P \gg m_{u,d}$
- Underlying physics non-perturbative
- Not restricted to QCD! Can also do other $SU(N_c)$ with fermions in different reps

What's currently the best lattice value for a particular quantity?

FLAG
Flavour Lattice Averaging Group

http://flag.unibe.ch

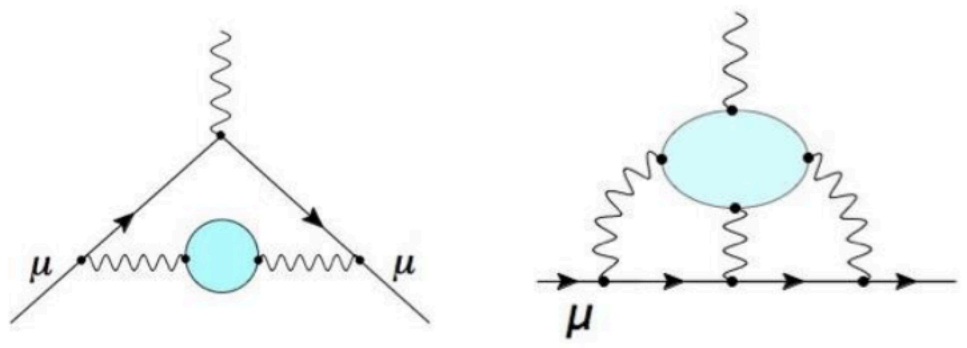
- summary of lattice results
- evaluation according to FLAG quality criteria
- averages or best values where possible (if sensible)
- detailed summary of properties of individual simulations
- target audience: wider phenomenology community

FLAG 11 Eur.Phys.J.C 71 (2011) 1695
FLAG 13 Eur.Phys.J.C 74 (2014) 2890
FLAG 16 Eur.Phys.J.C 77 (2017) 2, 112
FLAG 19 Eur.Phys.J.C 80 (2020) 2, 113
FLAG 21 Eur.Phys.J.C 82 (2022) 869
FLAG 24 arXiv:2411.04268

quark masses
strong coupling constant
bag parameters
nucleon matrix elements
meson leptonic decay
meson semilpetonic decay form factors
baryon semileptonic decay from factors
CKM matrix elements

Muon g-2 initiative

https://muon-gm2-theory.illinois.edu



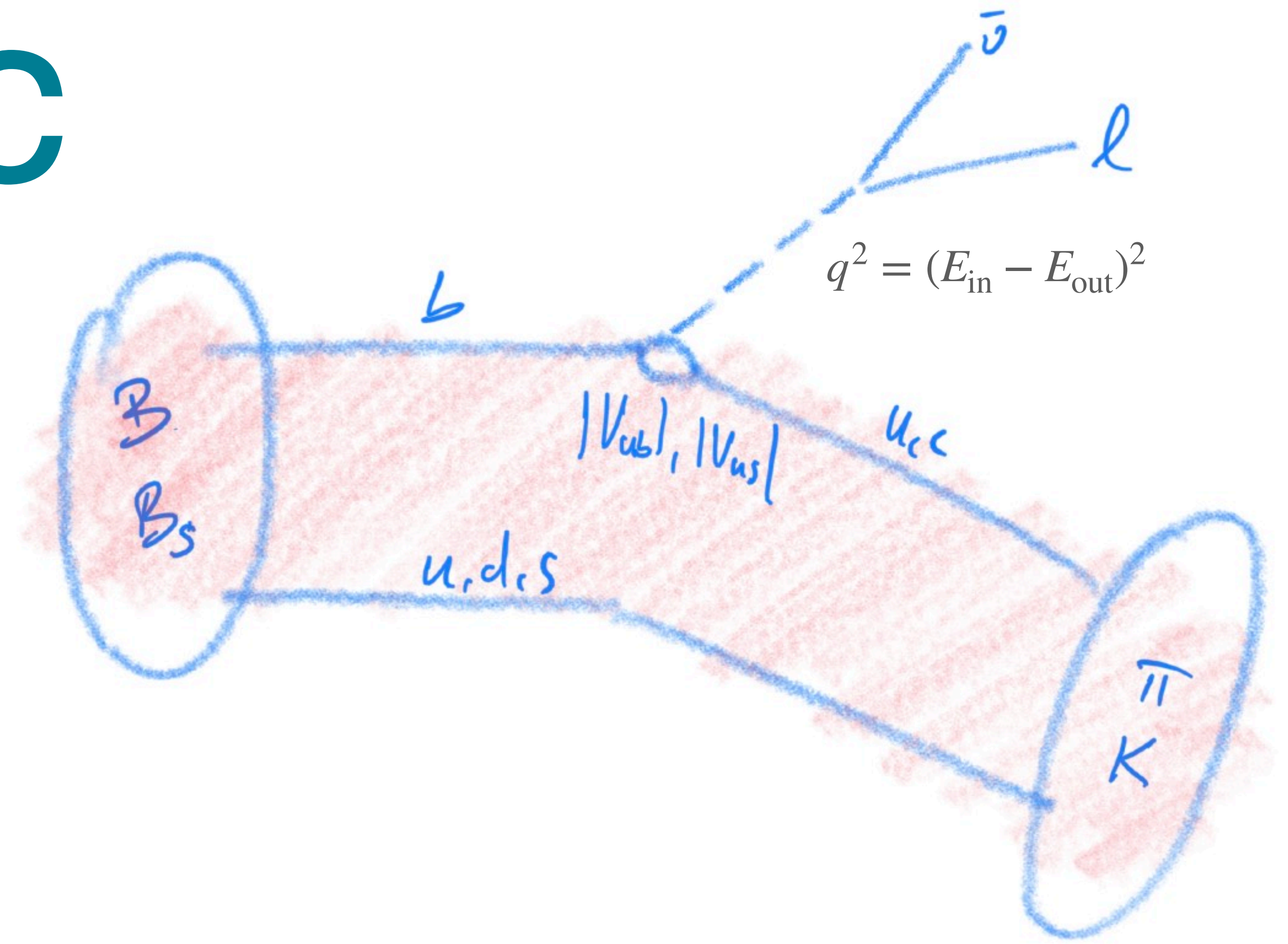
Phys.Rept. 887 (2020) 1-166

Part I:
QFT constraints for exclusive semileptonic meson decays

based on work in collaboration with

- **Jonathan Flynn (Southampton)** and **Tobi Tsang (CERN → Liverpool)** [JHEP (2023)]
- **Marzia Bordone (CERN/Zürich)** [EPJC (2025)]

Intro: exclusive semileptonic meson decay



Objective: obtain model-independent theory prediction over entire kinematical range

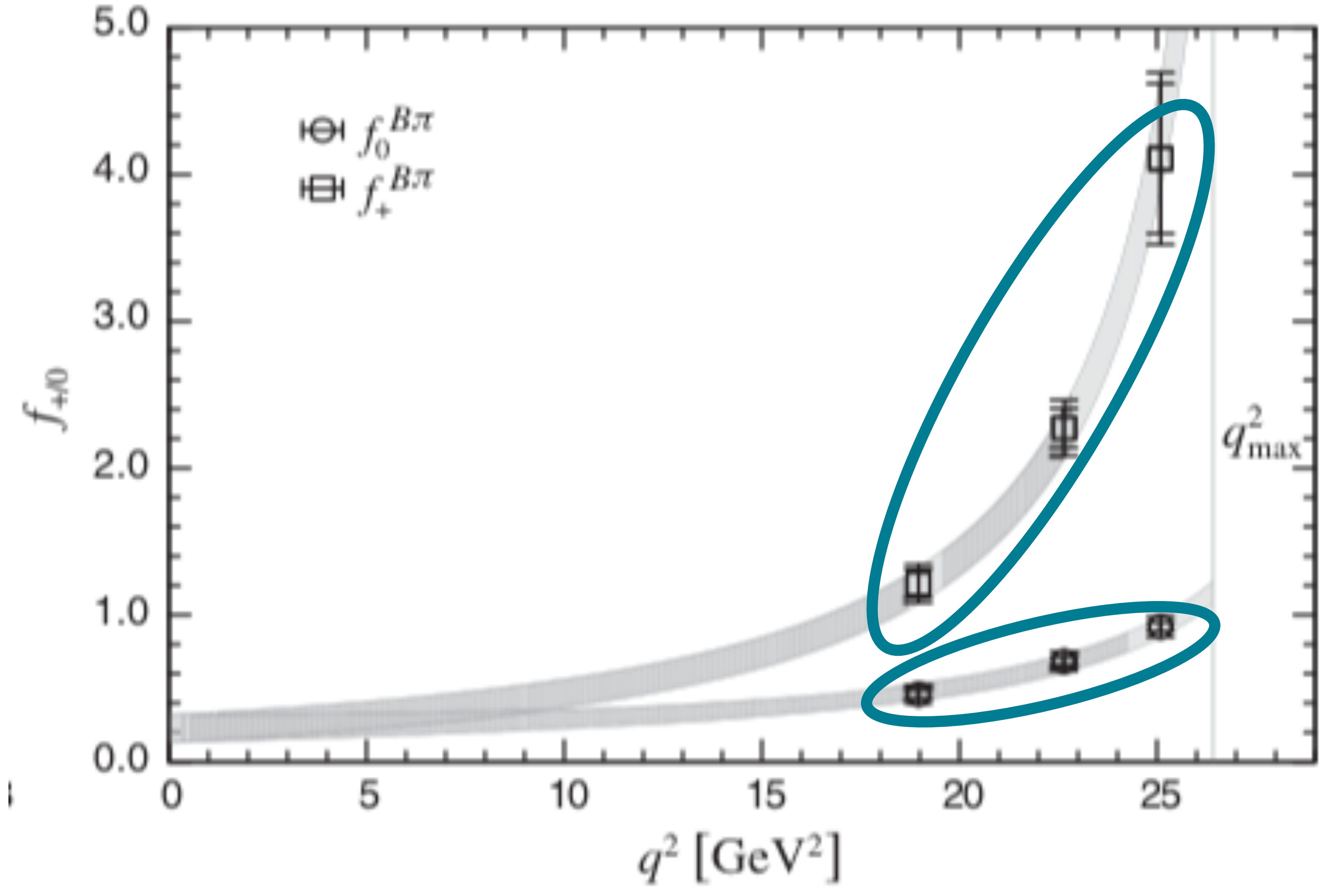
Input:

- sum rules: $q^2 \approx 0$
- lattice QCD:
    - finite lattice spacing (UV)
    - finite volume (IR)
    - worsening signal-to-noise

→ limited kinematic reach

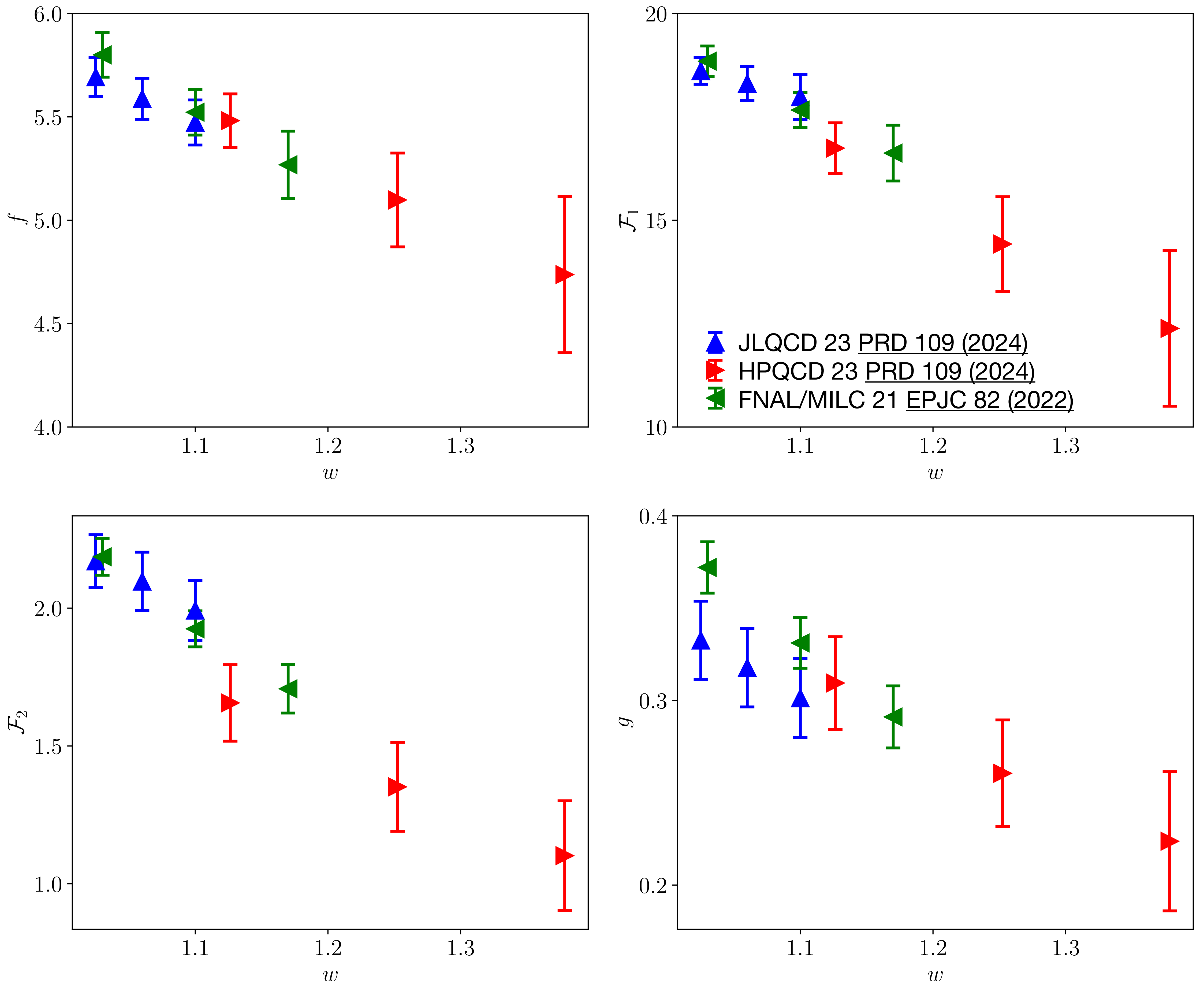
↓

need extra/interpolation



RBC/UKQCD PRD 91, 074510 (2015)

New lattice data – $B \to D^* \ell \bar{\nu}_\ell$



New lattice data

- four form factors $f, \mathscr{F}_1, \mathscr{F}_2, g$

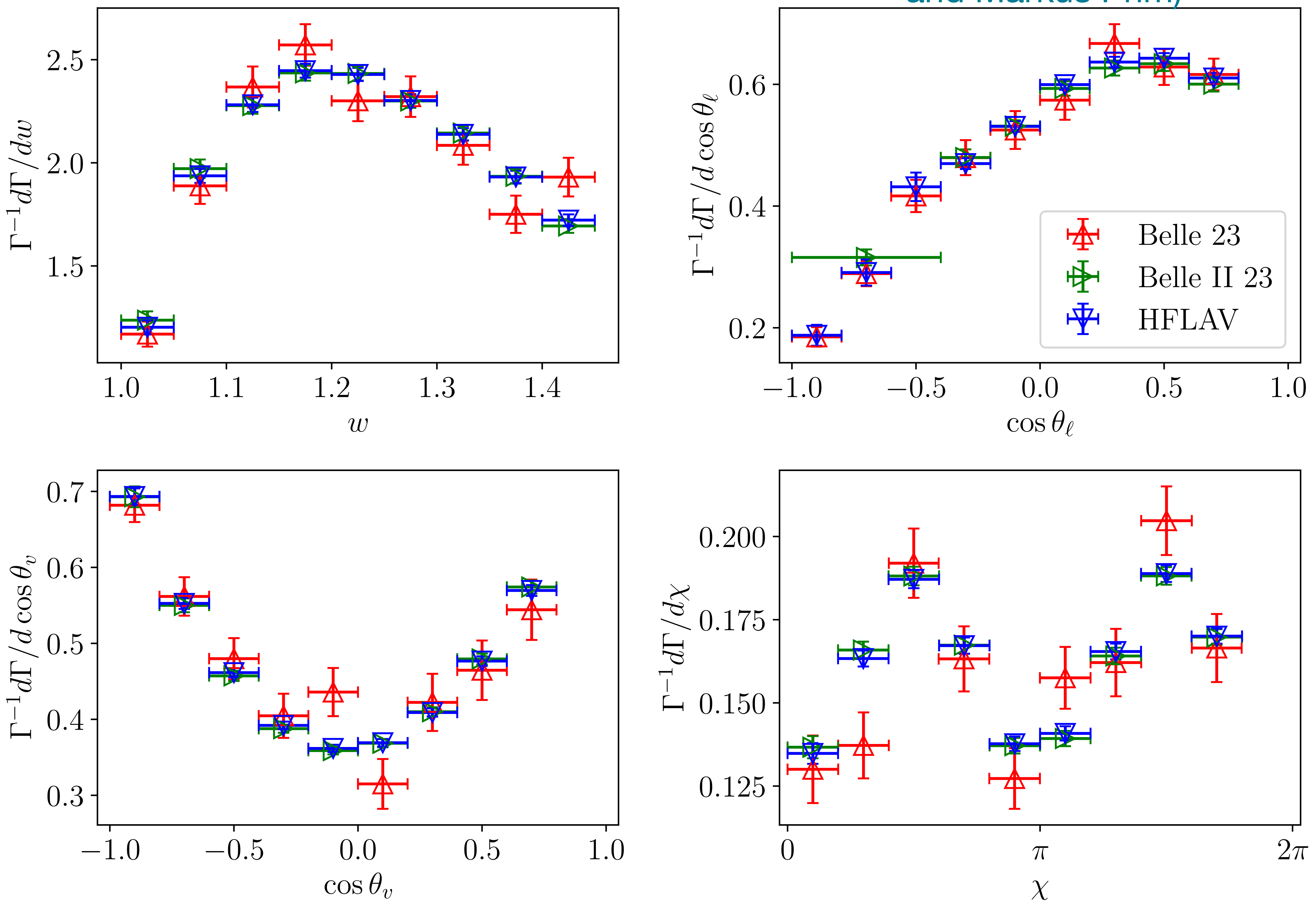
$$w = \frac{M_B^2 + M_{D*}^2 - q^2}{2 M_B M_{D*}} \qquad q_\mu = (p_B - p_{D*})_\mu$$

- first time that lattice data covers kinematical range
- three different and independent collaborations
- just in time for new experimental data ...

New experimental data – $B \to D^{*}\ell\bar{\nu}_{\ell}$

Belle Phys.Rev.D 108 (2023) 1, 012002
Belle II Phys.Rev.D 108 (2023) 9, 9

Thanks to HFLAV for combined data set (in particular Florian Bernlochner and Markus Prim)

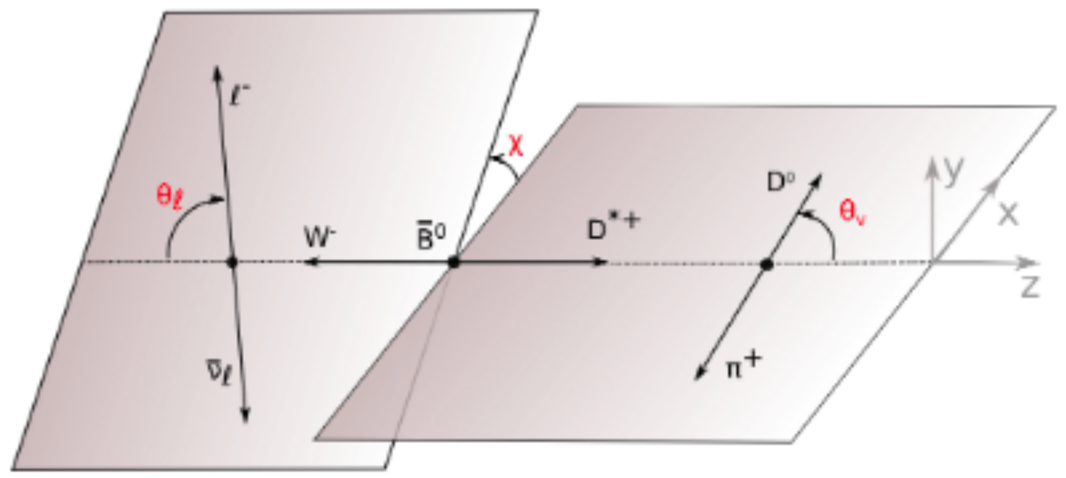


here not Belle 2310.20286 (angular coeffs)

New experimental data

- four (normalised) differential decay rates in channels

$$\alpha = w,\ \cos\theta_\ell,\ \cos\theta_v, \chi$$



- between 7 and 10 bins per α
- data available on HEPData
- two experimental collaborations
- just in time for new lattice data …

$$\frac{d\Gamma}{dw d\cos(\theta_\ell) d\cos(\theta_v) d\chi} = \frac{3G_F^2}{1024\pi^4}|V_{cb}|^2 \eta_{EW}^2 M_B r^2 \sqrt{w^2-1} q^2$$
$$\times \{(1-\cos(\theta_\ell))^2 \sin^2(\theta_v) H_+^2(w) + (1+\cos(\theta_\ell))^2 \sin^2(\theta_v) H_-^2(w)$$
$$+ 4\sin^2(\theta_\ell)\cos^2(\theta_v) H_0^2(w) - 2\sin^2(\theta_\ell)\sin^2(\theta_v)\cos(2\chi) H_+(w) H_-(w)$$
$$- 4\sin(\theta_\ell)(1-\cos(\theta_\ell))\sin(\theta_v)\cos(\theta_v)\cos(\chi) H_+(w) H_0(w)$$
$$+ 4\sin(\theta_\ell)(1+\cos(\theta_\ell))\sin(\theta_v)\cos(\theta_v)\cos(\chi) H_-(w) H_0(w)\}$$

How to best analyse this new quality of data as part of a precision test of the SM?

Fitting strategies — $|V_{cb}|, R(D^*)$ etc.

A

- fit parameterisation to lattice data
- compute theory prediction for $d\Gamma/dw/|V_{cb}|^2$ bin-by-bin by integration
- combine with experimental data for bin-by-bin prediction for $|V_{cb}|$
- final $|V_{cb}|$ from weighted average over bins

Clean separation of SM and exp. measurement

B

- fit parameterisation simultaneously to lattice form factors and results for experimental data for diff. decay rate (use shape-information from both experiment and lattice)
- determine $|V_{cb}|$ directly from such a global fit

Unitarity constraint and fit-ansatz imposed on experimental data (which may contain BSM)

C

- fit parameterisation only to measurements of normalised diff. decay rate (use shape-information from experiment only) (need theory normalisation for CKM prediction)

Related work

Bordone, AJ, EPJC 85 (2025) 2, 129
Fedele et al. PRD 108 (2023) 5, 5
Flynn, AJ, Tsang JHEP 12 (2023)

Martinelli et al. EPJC 85 (2025) 3, 242
PRD 111 (2025) 1, 013005
EPJC 84 (2024) 4, 400,
PRD 106 (2022) 9, 093002,
EPJC 82 (2022) 12,
PRD 105 (2022) 3, 034503,
PRD 104 (2021) 9, 094512
Di Carlo et al. PRD 104 (2021) 5, 054502

Ray, Nandi JHEP 01 (2024) 022
Gambino PLB 795 (2019) 386-390
Bigi PLB 769 (2017) 441-445, JHEP 11 (2017) 061
Bernlochner et al. PRD 100 (2019) 1, 013005

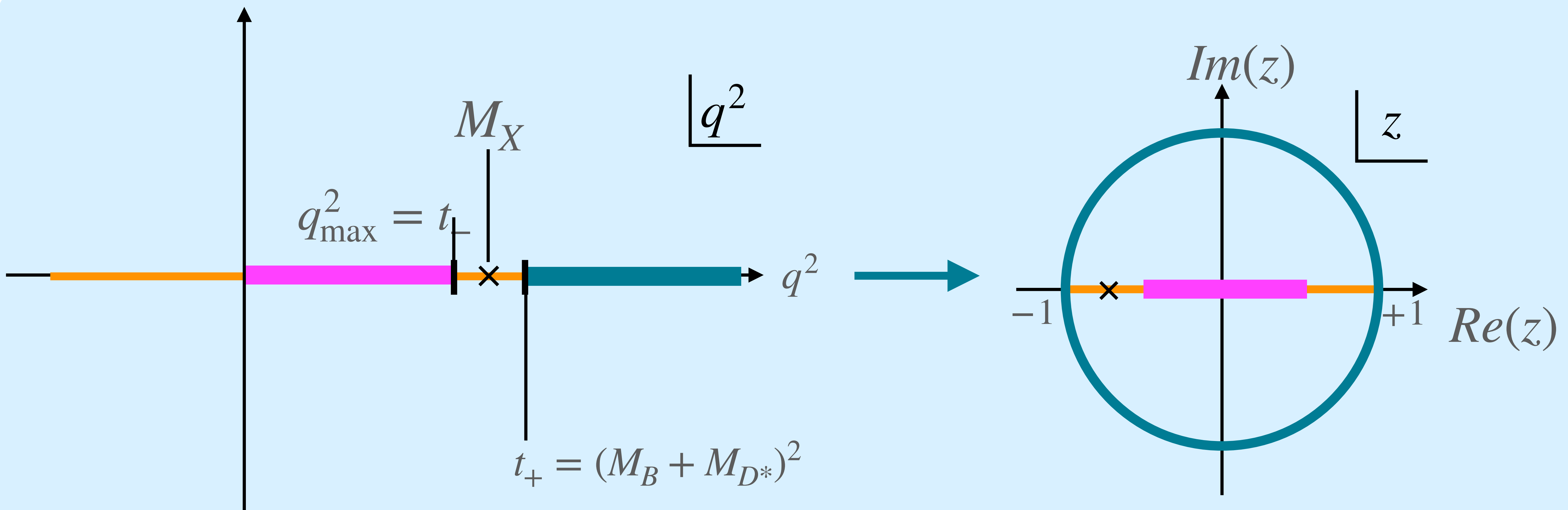
SM correct — A, B and C should result in compatible predictions

Form-factor parameterisation

Boyd-Grinstein-Lebed ansatz:

$$f_X(q_i^2) = \frac{1}{B_X(q_i^2)\phi_X(q_i^2, t_0)} \sum_{n=0}^{\infty} a_{X,n} z(q_i^2)^n$$

Boyd, Grinstein, Lebed, PRL 74 (1995)

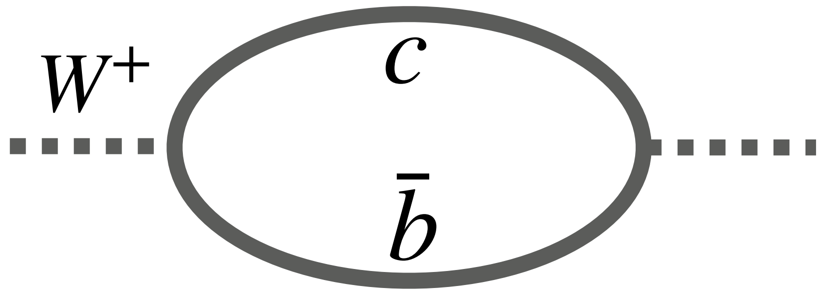


$z \in [0,0.06]$ for $B \to D^* \ell \bar{\nu}_\ell$

- model-independent form-factor parameterisation
- infinite # of parameters $a_{X,i}$
- finite # of data points → require truncation

[Okubo, PRD 3, 2807 (1971), PRD 4, 725 (1971)]
[Okubo, Shih, PRD 4, 2020 (1971)]
[Boyd, Grinstein, Lebed, PLB 353, 306 (1995), NPB461, 493 (1996). PRD 56, 6895 (1997)]

QFT constraints



VP tensor: $$\Pi_V^{\mu\nu}(q) = i\int d^4x e^{iqx}\langle 0|T\{V^\mu(x)V^{\nu\dagger}(0)|0\rangle = \frac{1}{q^2}(q^\mu q^\nu - q^2 g^{\mu\nu})\Pi_{1^-}(q^2) + \frac{q^\mu q^\nu}{q^2}\Pi_{0^+}(q^2)$$

disp. relations: $$\chi_{1^-}(q^2) = \frac{1}{\pi}\int_0^\infty dt \frac{t\mathrm{Im}\Pi_{1^-}(t)}{(t-q^2)^3} \qquad \chi_{0^+}(q^2) = \frac{1}{\pi}\int_0^\infty dt \frac{t\mathrm{Im}\Pi_{0^+}(t)}{(t-q^2)^2}$$

spectral sum: $$\mathrm{Im}\Pi_V(q^2) \sim \frac{1}{2}\sum_X (2\pi)^4\delta^4(q-p_X)|\langle 0|V|X\rangle|^2$$

e.g. $X = BD^*$: $|\langle 0|V|BD^*\rangle| \propto |\langle D^*|V|B\rangle|$

unitarity constraint: $$\frac{1}{\pi\chi_J(q^2)}\int_{t_+}^\infty dt \frac{W(t)|f_J(t)|^2}{(t-q^2)^n} \leq 1$$

- dispersion relation leads to constraint on form factor in each symmetry channel
- χ can be evaluated in perturbation theory, e.g. at $q^2 = 0$ (or lattice [Martinelli et al. PR 104 (2021) 094512])

Form-factor parameterisation

$$f_X(q_i^2) = \frac{1}{B_X(q_i^2)\phi_X(q_i^2, t_0)} \sum_{n=0}^{\infty} a_{X,n} z(q_i^2)^n$$

dispersion relation + BGL ansatz:

$$\frac{1}{\pi\chi_X(q^2)} \int_{t_+}^{\infty} dt \frac{tW(t)\,|f_X(t)|^2}{(t-q^2)^n} \leq 1 \longrightarrow \frac{1}{2\pi i}\oint_C \frac{dz}{z} |B_X(q^2)\phi_X(q^2, t_0) f_X(q^2)|^2 \leq 1$$

$$\downarrow$$

$$|\mathbf{a}_X|^2 \leq 1$$

unitarity constraint
on BGL coefficients

Form-factor parameterisation

$$f_X(q_i^2) = \frac{1}{B_X(q_i^2)\phi_X(q_i^2, t_0)} \sum_{n=0}^{K_X-1} a_{X,n} z(q_i^2)^n$$

unitarity constraint: $|\mathbf{a}_X|^2 \leq 1$

Boyd, Grinstein, Lebed, PRL 74 (1995)

Determine all $a_{X,n}$ from finite set of theory data

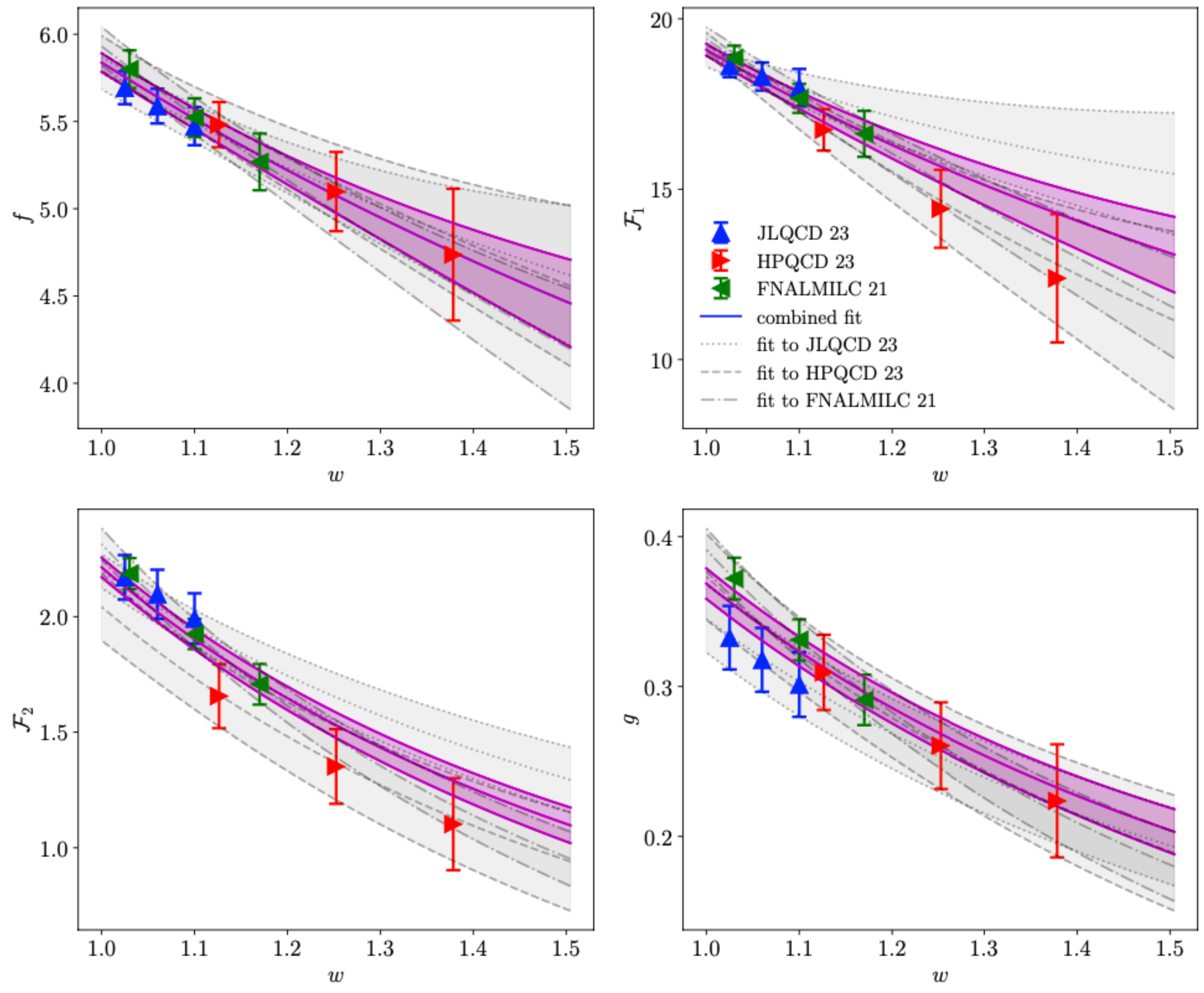
Frequentist fit:
- $N_{\mathrm{dof}} = N_{\mathrm{data}} - K_X \geq 1$
  $\rightarrow$ in practice truncation K at low order
- induced systematic difficult to estimate
- meaning of Frequentist with unitarity constraint?

Bayesian fit:
- fit including higher order z expansion meaningful
- unitarity regulates and controls higher-order coefficients [Flynn, AJ, Tsang JHEP 12 (2023) 175]
- well-defined meaning of unitarity constraint

Recommendation: Combined Frequentist + Bayesian perspective

Strategy A: Fit to lattice data

[Bordone, AJ, EPJC (2025)]



Frequentist fit

K_f	K_{F_1}	K_{F_2}	K_g	$a_{g,0}$	$a_{g,1}$	$a_{g,2}$	$a_{g,3}$	p	χ^2/N_{dof}	N_{dof}
2	2	2	2	0.03138(87)	-0.059(24)	-	-	0.95	0.62	30
3	3	3	3	0.03131(87)	-0.046(36)	-1.2(1.8)	-	0.90	0.67	26
4	4	4	4	0.03126(87)	-0.017(48)	-3.7(3.3)	49.9(53.6)	0.79	0.75	22

- good fit quality
- lattice data compatible
- no unitarity constraint

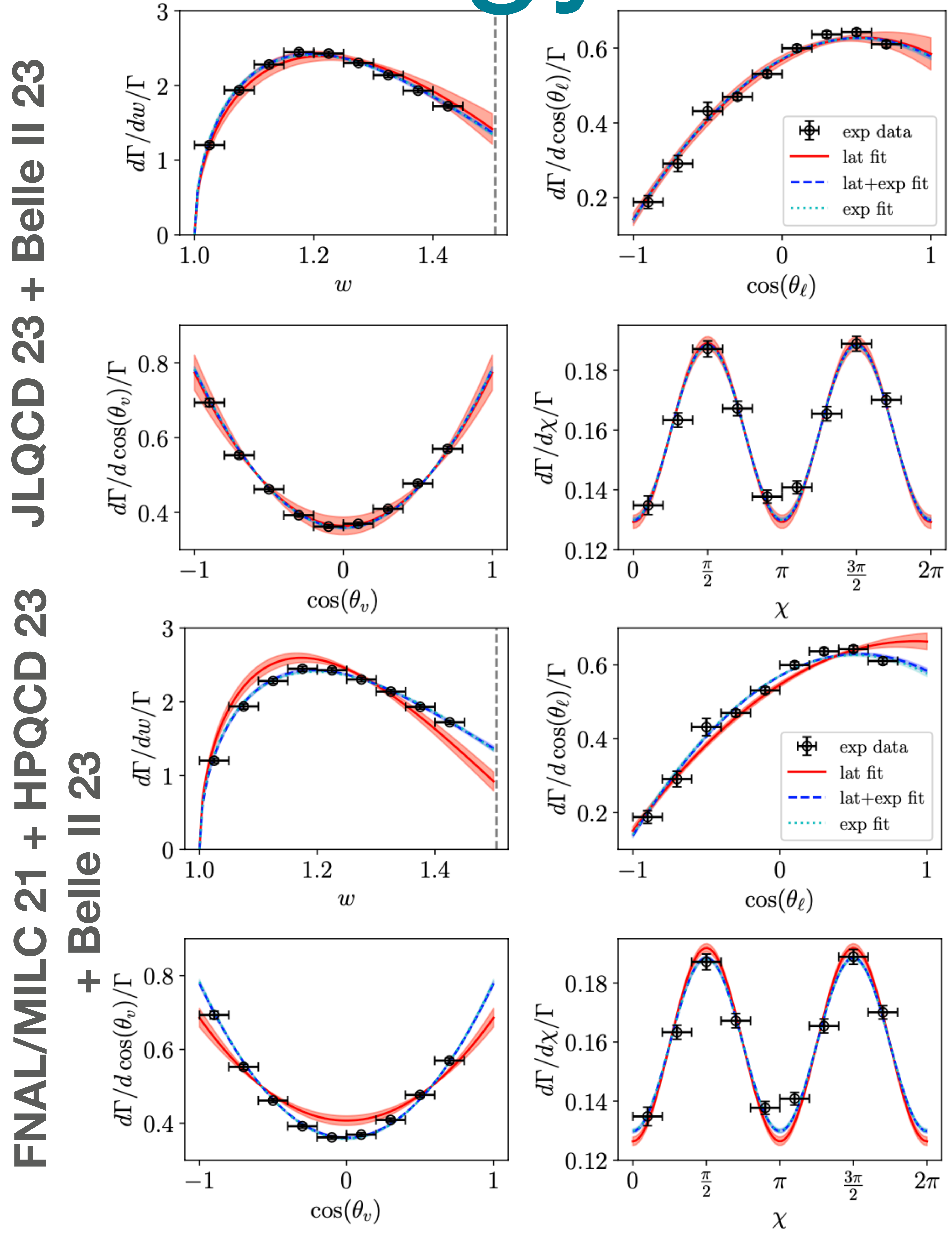
Bayesian inference

K_f	K_{F_1}	K_{F_2}	K_g	$a_{g,0}$	$a_{g,1}$	$a_{g,2}$	$a_{g,3}$
2	2	2	2	0.03133(80)	-0.058(25)	-	-
3	3	3	3	0.03129(81)	-0.062(27)	-0.10(55)	-
4	4	4	4	0.03134(86)	-0.061(25)	-0.10(50)	-0.04(49)

- unitarity constraint regulates higher-order coefficients
- truncation independent

Strategy B: Fit to lattice + exp.data

Bordone, AJ, EPJC (2025)



- BGL fit to only lattice data (strategy A) misses experimental points for two of the lattice data sets
- BGL fit to experimental and lattice data of good quality

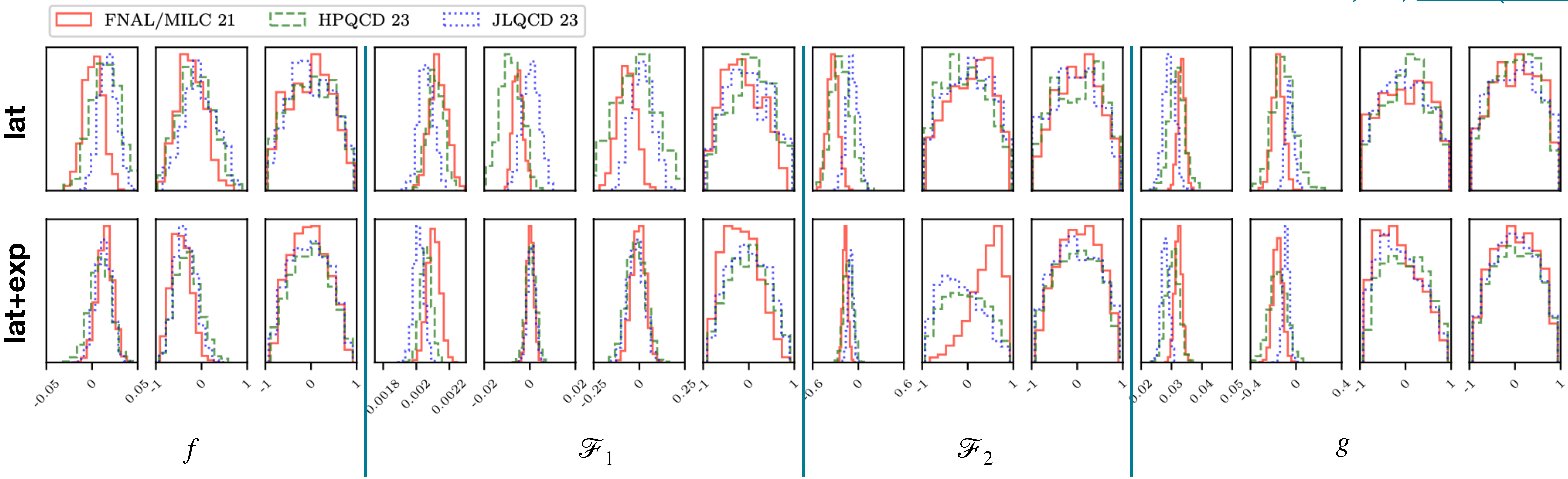
  Frequentist fit quality good

  lat $(p, \chi^2/N_{\mathrm{dof}}, N_{\mathrm{dof}}) = (0.79, 0.75, 22)$

  lat+exp $(p, \chi^2/N_{\mathrm{dof}}, N_{\mathrm{dof}}) = (0.18, 1.15, 56)$
- some BGL coefficients shift between strategy A) and B) by up to a few σ → but precision of lattice data allows for enough wiggle room
- BGL fit to only experimental data (strategy C)

Strategy B: Fit to lattice + experimental data

Bordone, AJ, EPJC (2025)



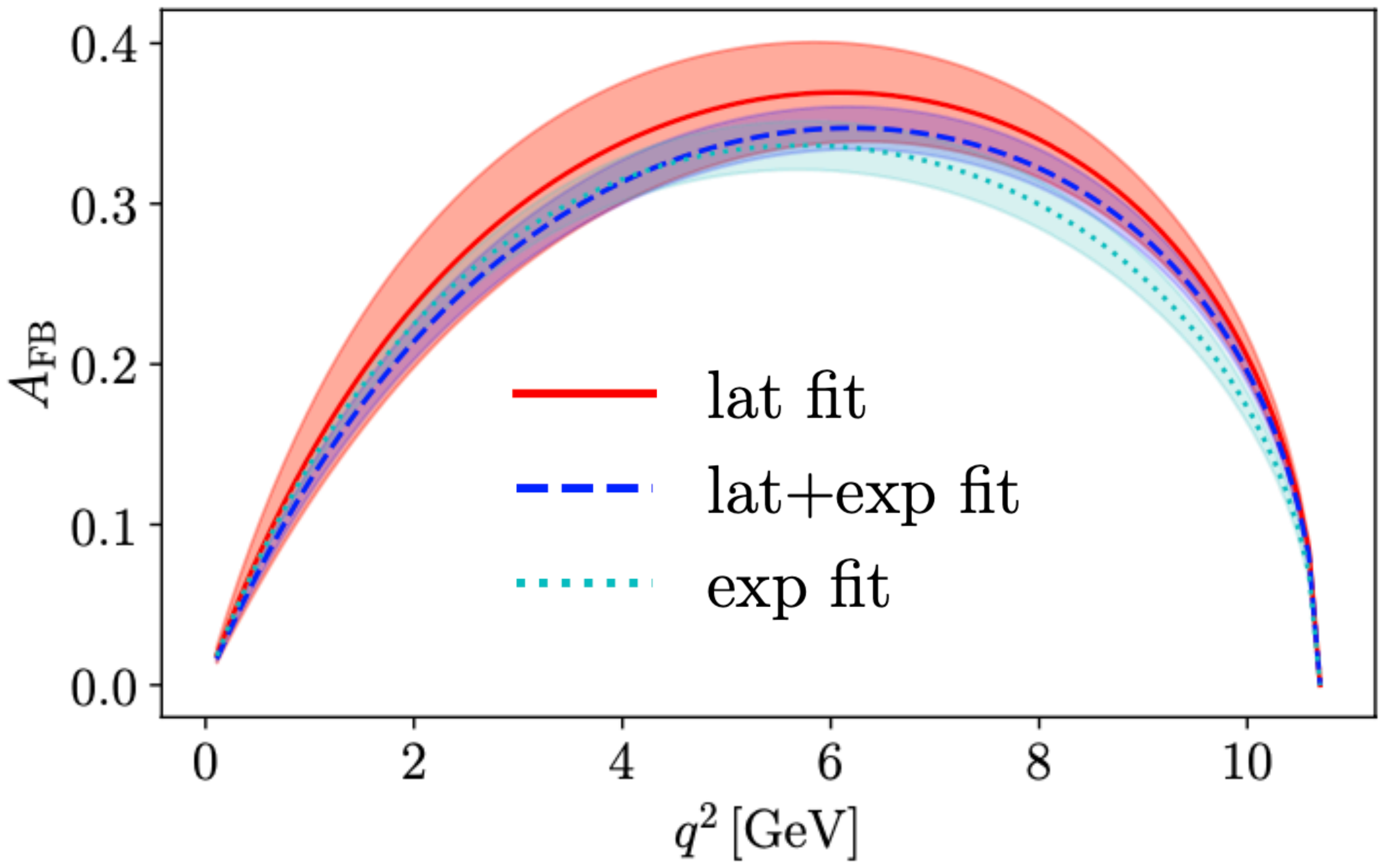
- posterior distribution reflect small shifts between lat and lat+exp fits
- higher-order coefficients “regulated” by unitarity constraint
- strange behaviour in $a_{\mathscr{F}_{2,2}}$ for FNAL/MILC-based fit?

Other observables

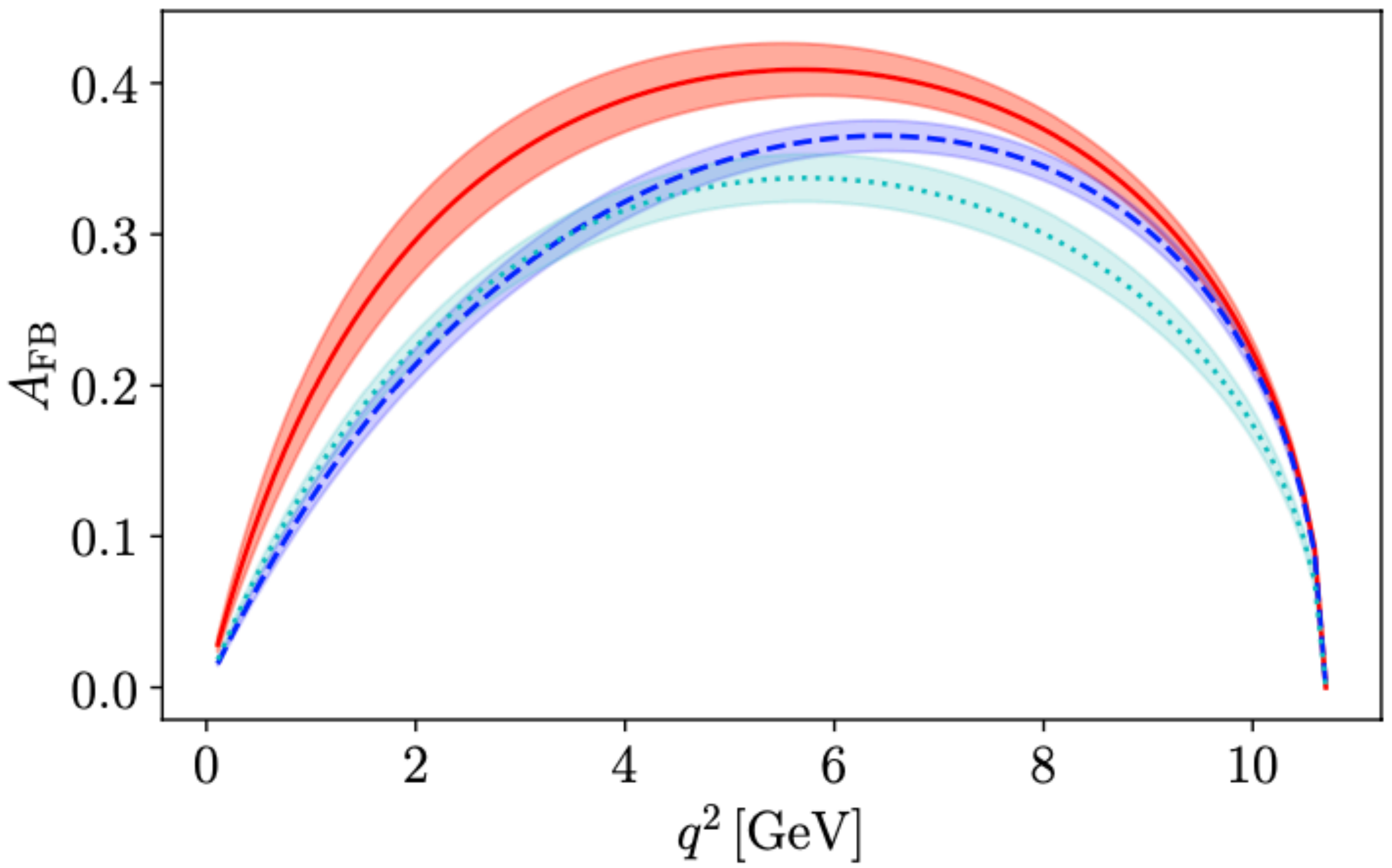
Bordone, AJ, EPJC (2025)

e.g. Forward-Backward asymmetry
$$A_{\mathrm{FB}} = \frac{\int_0^1 - \int_{-1}^0 d\cos\theta_\ell d\Gamma/d\cos\theta_\ell}{\int_0^1 + \int_{-1}^0 d\cos\theta_\ell d\Gamma/d\cos\theta_\ell}$$

JLQCD 23 + Belle II 23

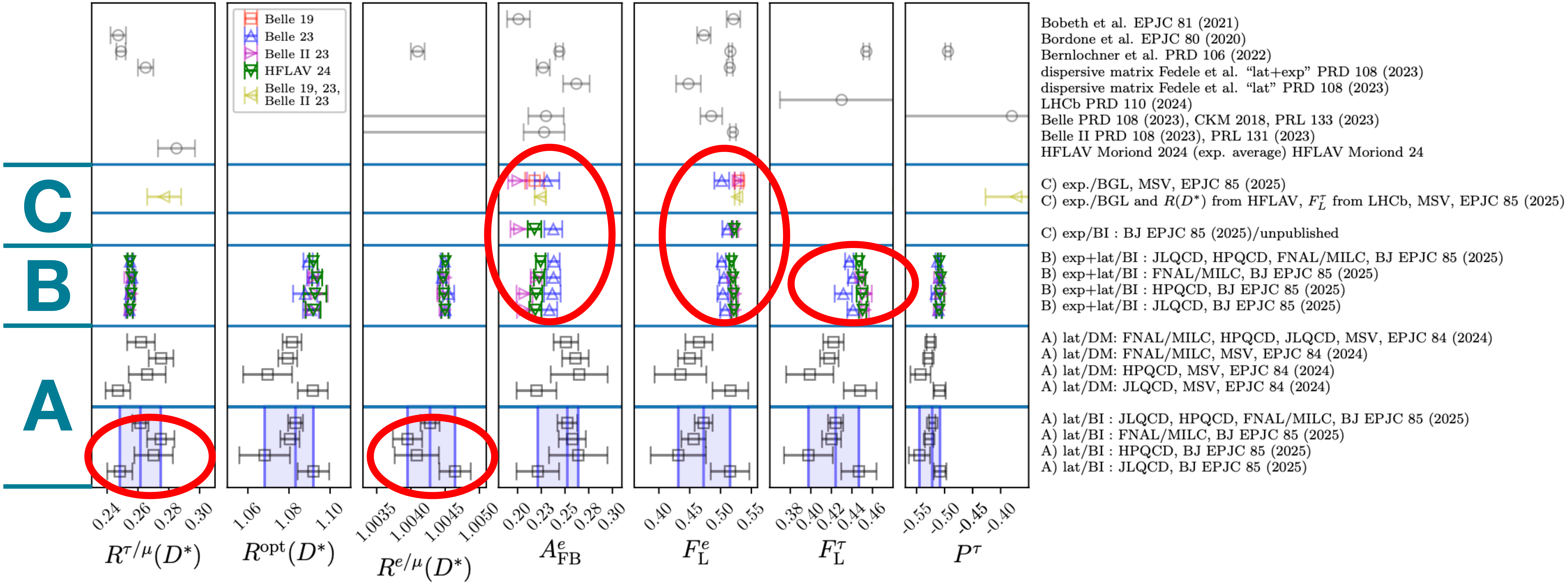


FNAL/MILC 21 + HPQCD 23 + Belle II 23



precision of lat and exp data allow to identify differences in shapes of distributions
Here: lat, lat+exp and exp parameterisations exhibit distinctly different shapes

Other observables



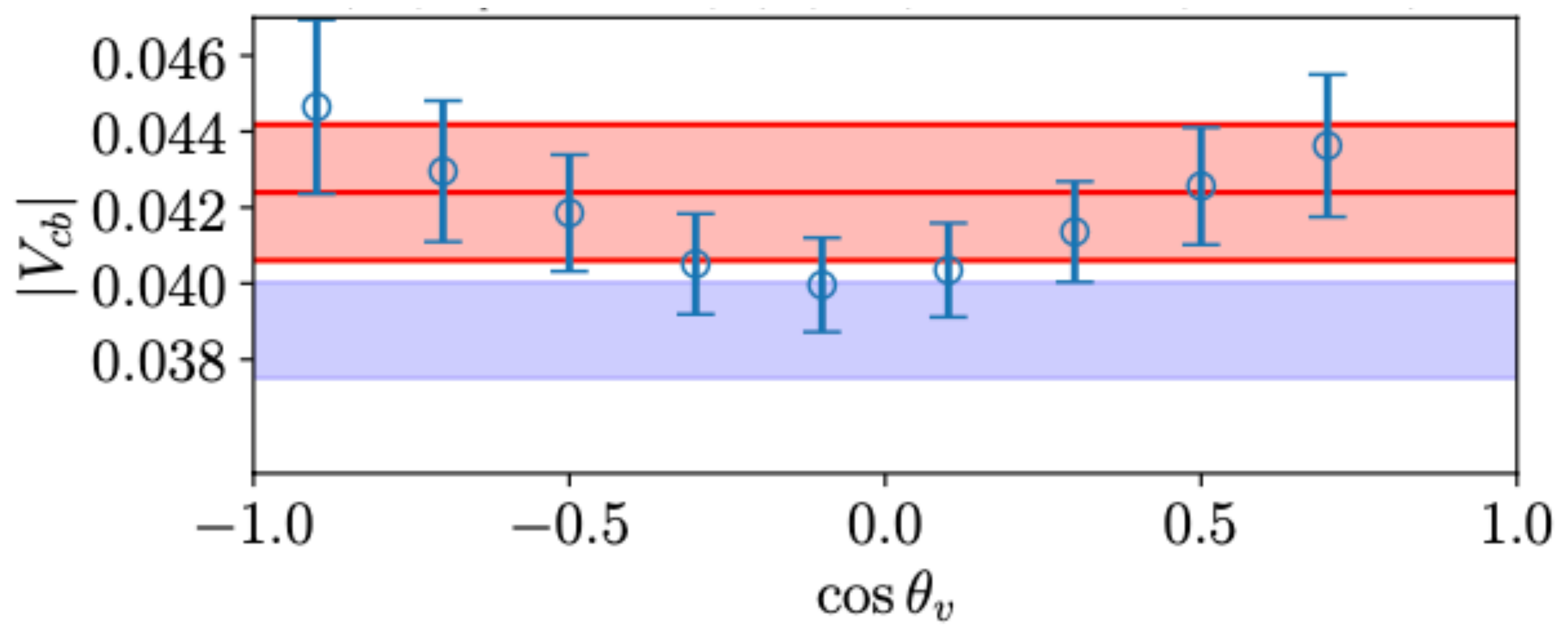
- **A) lat (DM,BI): tensions amongst results from different lattice collaborations**
- **B) lat+exp (BI):** lattice consistent but driven by exp., **tensions amongst experiments**
- **C) exp (BGL,BI): tension amongst experiments**

analysis reveals tensions amongst lattice as well as amongst experimental

$|V_{cb}|$ — Strategy A: Fit to lattice data

Bordone, AJ, 2406.10074

$$|V_{cb}|_{\alpha,i} = \left(\Gamma_{\rm exp}\left[\frac{1}{\Gamma}\frac{d\Gamma}{d\alpha}\right]^{(i)}_{\rm exp} \Big/ \left[\frac{d\Gamma_0}{d\alpha}(\mathbf{a})\right]^{(i)}_{\rm lat}\right)^{1/2}, \qquad \text{where} \qquad \Gamma_{\rm exp} = \frac{\mathscr{B}(B^0 \to D^{*,-}\ell^+\nu_\ell)}{\tau(B^0)},$$



blue:
- Frequentist fit $(p, \chi^2/N_{\rm dof}, N_{\rm dof}) = (0.00, 2.82, 8)$
- d'Agostini Bias? [d'Agostini, Nucl.Instrum.Meth.A 346 (1994)]

red:
- Akaike-Information-Criterion analysis [H. Akaike IEEE TAC (19,6,1974)] average over all possible fits with at least two data points and then weighted average:

$$w_{\{\alpha,i\}} = \mathcal{N}^{-1}\exp\left(-\frac{1}{2}(\chi^2_{\{\alpha,i\}} - 2N_{{\rm dof},\{\alpha,i\}})\right) \qquad \mathcal{N} = \sum_{{\rm set}\in\{\alpha,i\}} w_{\rm set}$$

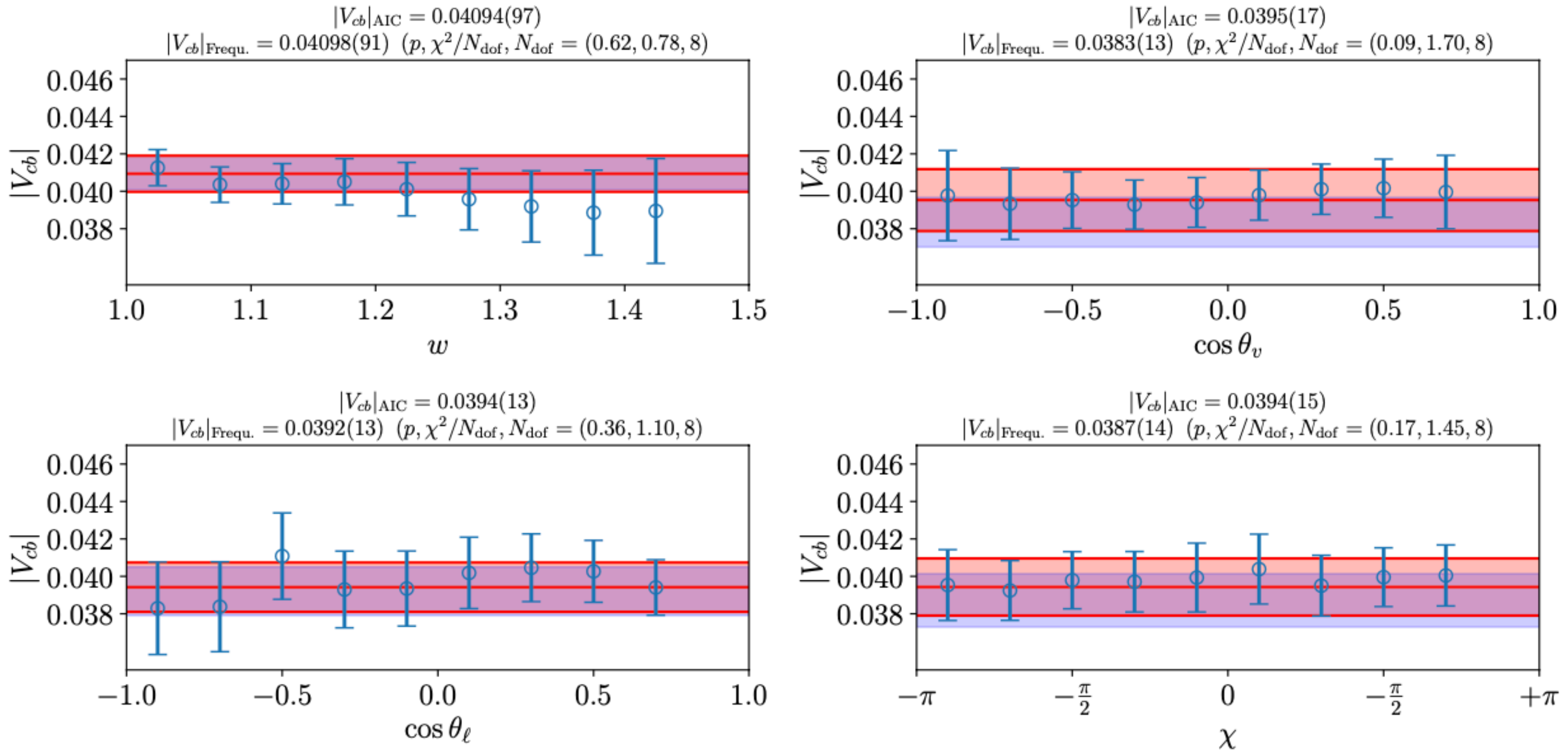
$$|V_{cb}| = \langle |V_{cb}| \rangle \equiv \sum_{{\rm set}\in\{\alpha,i\}} w_{\rm set}|V_{cb}|_{\rm set}$$

- result more *sensible* and bias apparently reduced

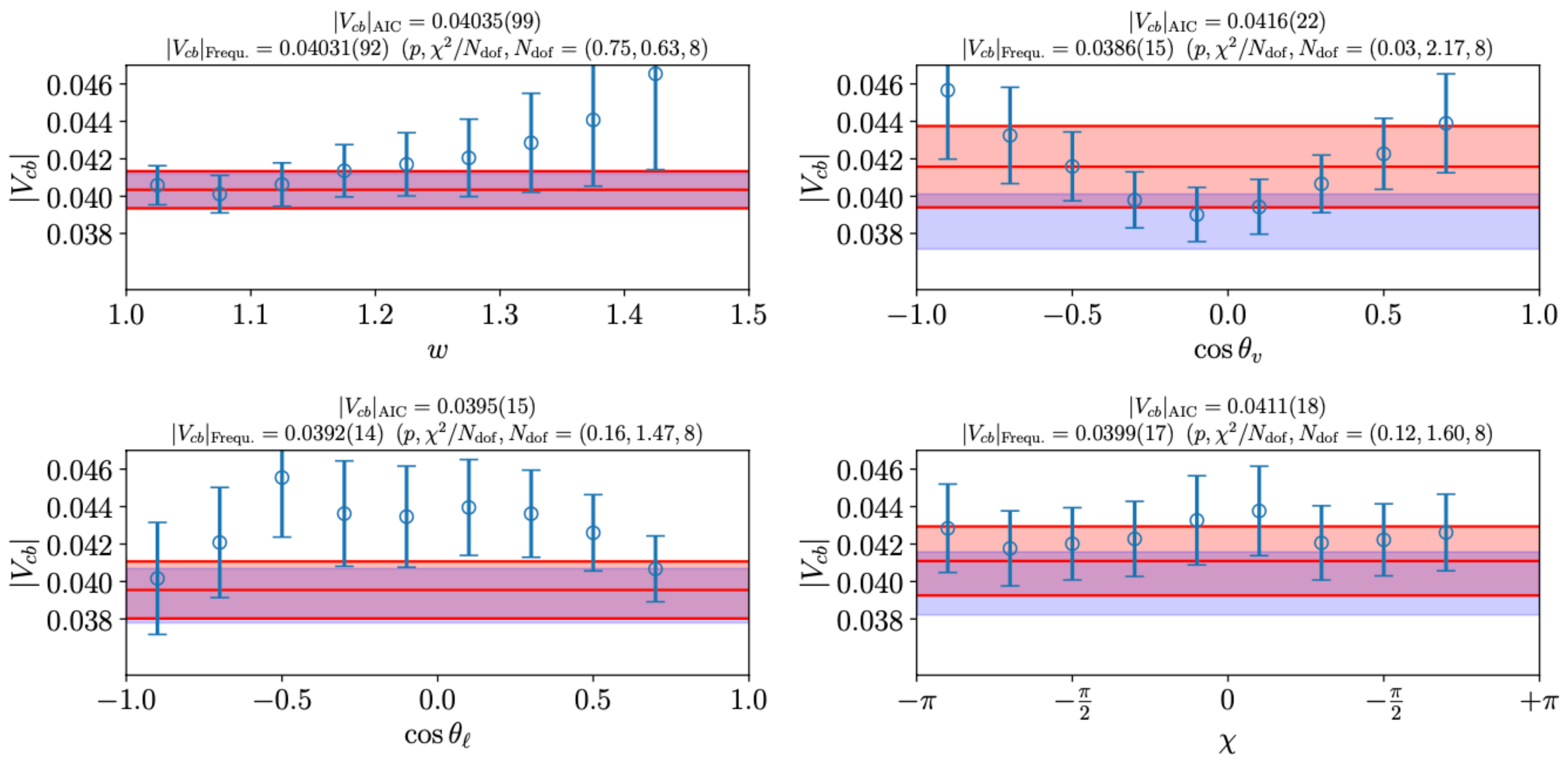
$|V_{cb}|$ — Strategy A: different lattice input — different

Bordone, AJ, [2406.10074](2406.10074)

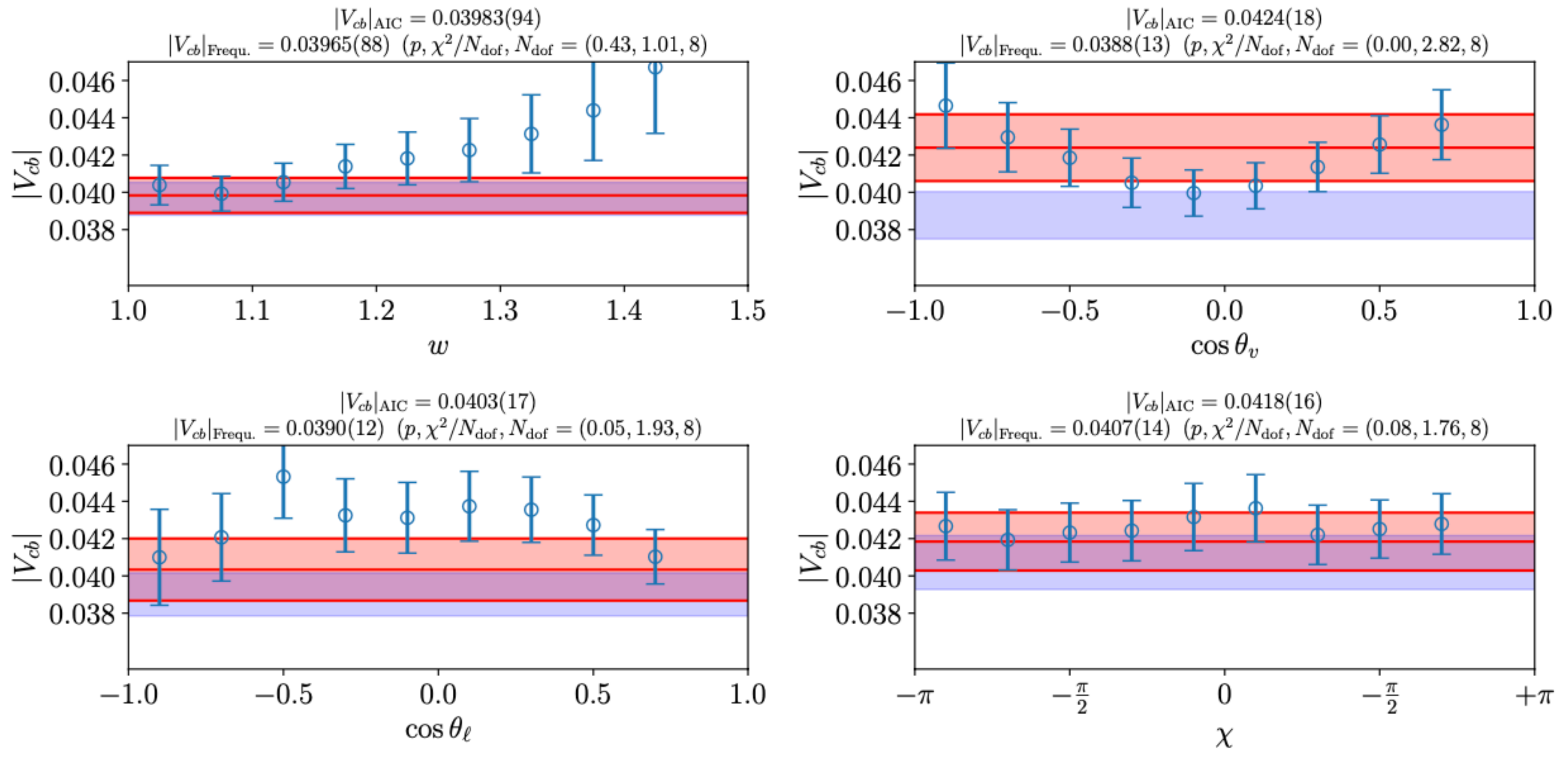
JLQCD 23



HPQCD 23

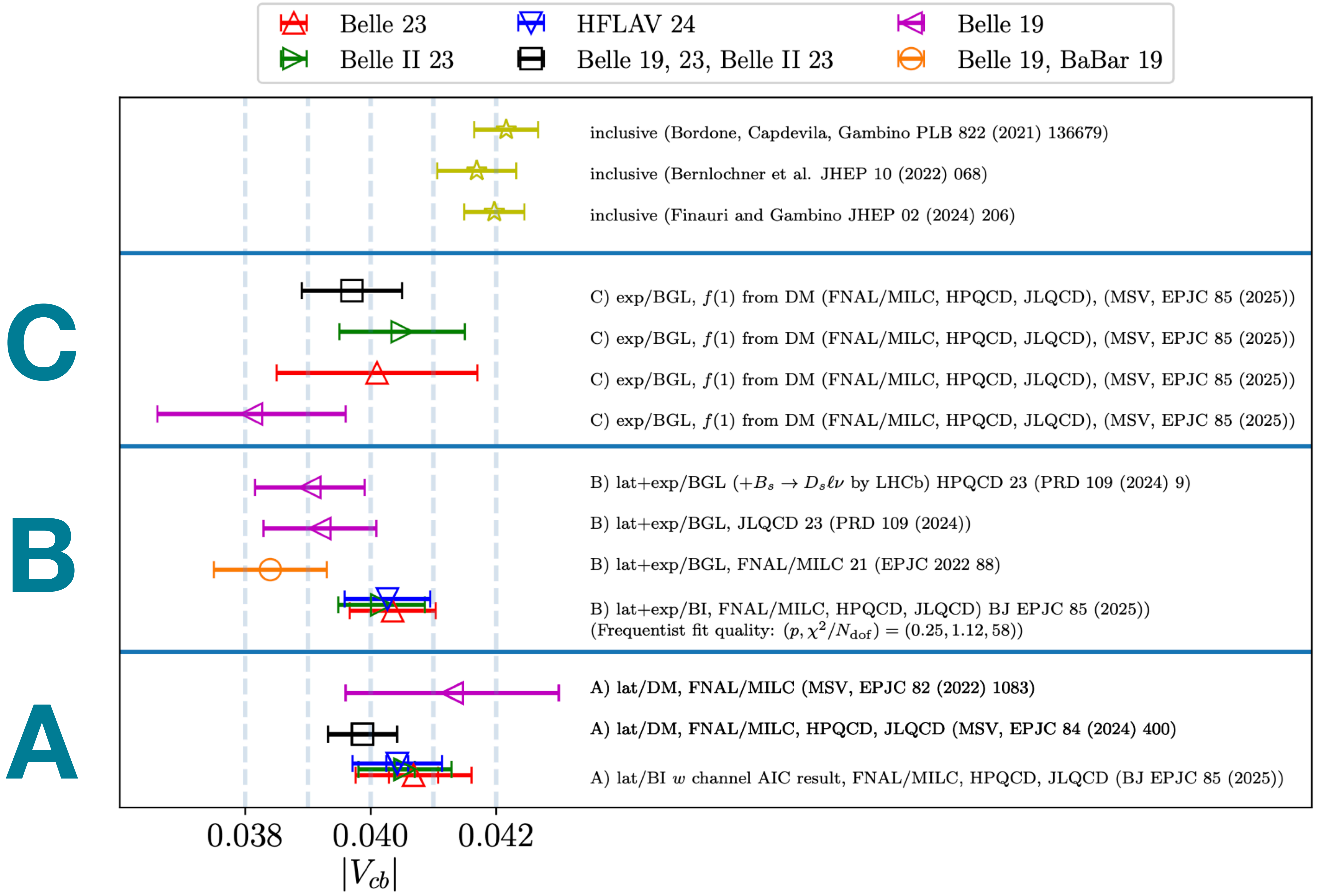


FNAL/MILC 21



- AIC approach works nicely
- some lattice data however problematic and at odds with expectation
- in particular analysis of angular distributions problematic?
- discard analysis $X = \cos\theta_v, \cos\theta_\ell, \chi$?

$|V_{cb}|$ — Summary



strategy A) BGL fit to lattice data, then combination with experiment

strategy B) BGL fit to both lattice and experiment

strategy C) BGL fit to experiments only, then CKM matrix element inferred from total BR

- exclusive analysis tends to come out lower than inclusive
- analyses based on Belle 19 generally lower than Belle (II) 23

Summary — Part I

Framework for truncation- and model-independent form-factor fitting
combining Frequentist and Bayesian statistics

Bayesian inference ansatz:

- Shown for $P \to P$ and $P \to V$ transitions
- Framework imposes unitarity constraint with meaningful statistical interpretation
- Unitarity constraint acts as regulator for higher-order coefficients
- Results converge to stable and truncation-independent values

- New quality of $B \to D^* \ell \bar{\nu}_\ell$ data from theory and experiment
- SM tests passed at current level of precision
- but our analysis shows tensions amongst lattice as well as experimental data sets
- $|V_{cb}|$ tension between inclusive and exclusive remains a puzzle

Also have a look:

- Dispersive-matrix method, Di Carlo et al. PRD 2021, arXiv:2105.02497
- Self-consistency checks of z expansinon, Simons, Gustafson, Meurice arXiv:2304.13045

Part II:
QFT constraints for inclusive meson decays (on the lattice)

ongoing work in collaboration with
Alessandro Barone (Mainz)
Ahmed Elgazhari (Soton)
Shoji Hashimoto (KEK)
Takashi Kaneko (KEK)
Ryan Kellermann (KEK)
Hu Zhi (KEK)

JHEP 07 (2023) 145

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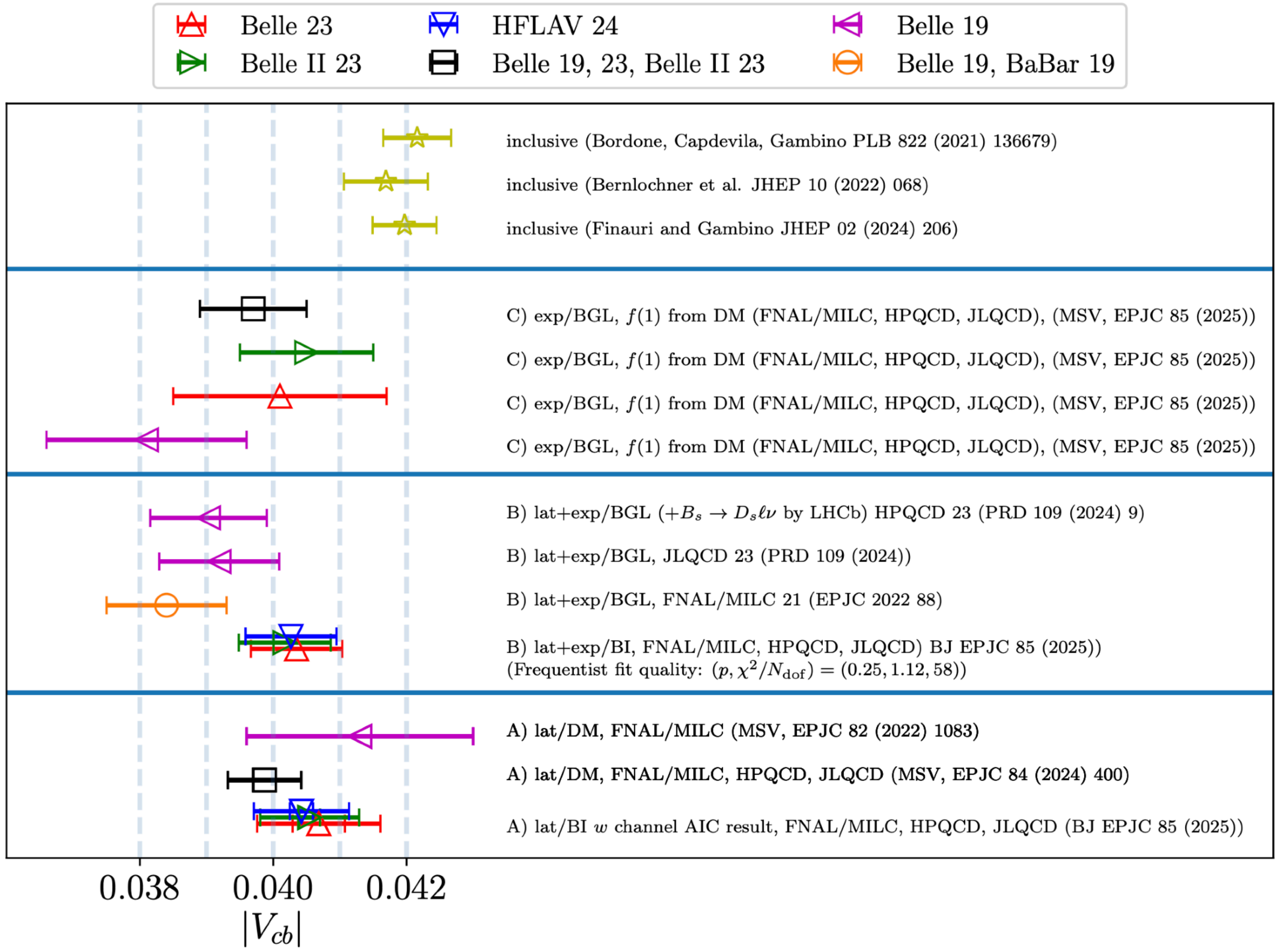
arXiv:2504.03358

A puzzle — inclusive vs. exclusive SL decay



exclusive decays — well understood (really?)

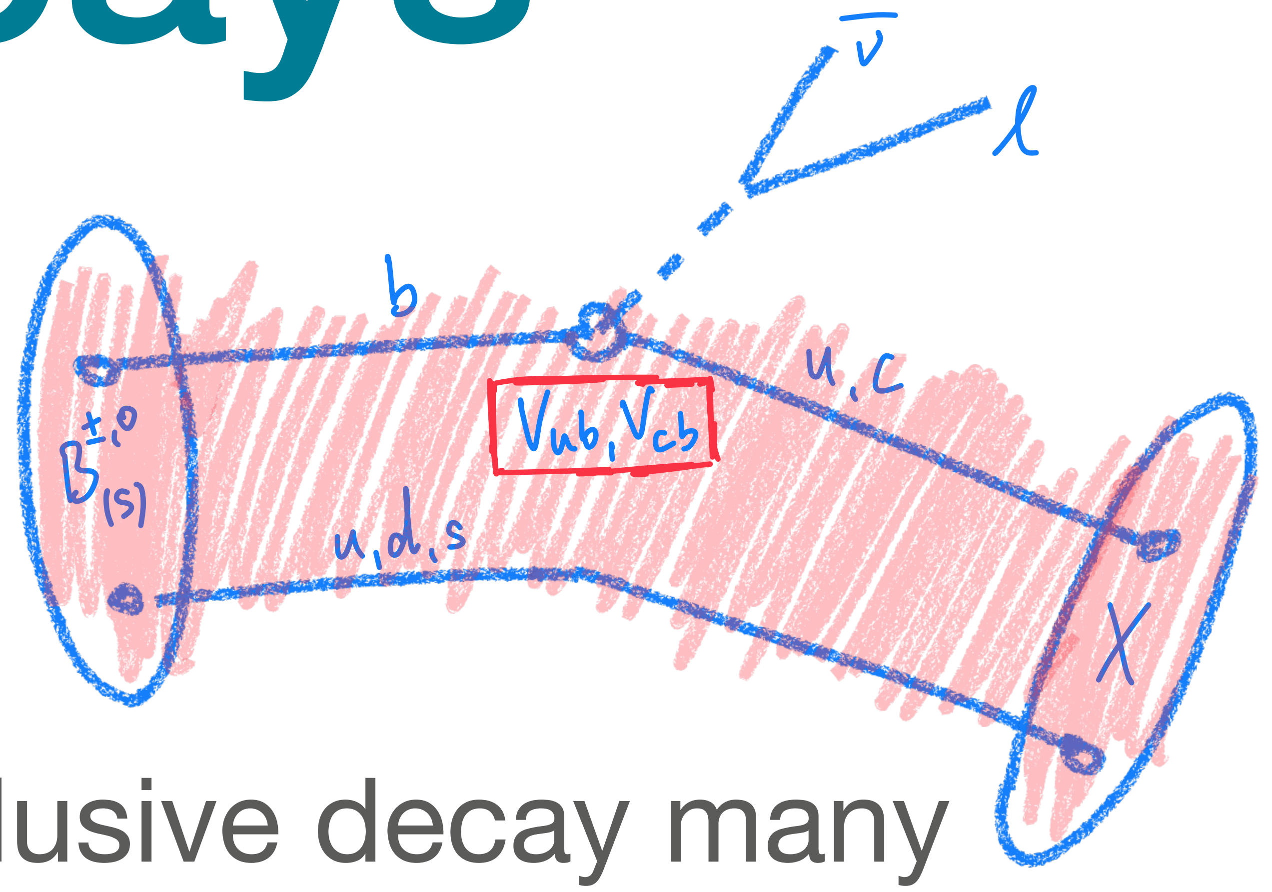
inclusive decays — to date no complete lattice computation

Inclusive SL decays

$B^0 \to X_c \ell^+ \nu_l$

Semileptonic and leptonic modes **PDGlive**

Γ_1	$\ell^+\nu_\ell X$	[1]	$(10.99 \pm 0.28)\%$		
Γ_2	$e^+\nu_e X_c$		$(10.8 \pm 0.4)\%$		
Γ_3	$D\ell^+\nu_\ell X$		$(9.7 \pm 0.7)\%$		
Γ_4	$\overline{D}^0\ell^+\nu_\ell$	[1]	$(2.35 \pm 0.09)\%$		2310
Γ_5	$\overline{D}^0\tau^+\nu_\tau$		$(7.7 \pm 2.5) \times 10^{-3}$		1911
Γ_6	$\overline{D}^*(2007)^0\ell^+\nu_\ell$	[1]	$(5.66 \pm 0.22)\%$		2258
Γ_7	$\overline{D}^*(2007)^0\tau^+\nu_\tau$		$(1.88 \pm 0.20)\%$	S=1.0	1839
Γ_8	$D^-\pi^+\ell^+\nu_\ell$		$(4.4 \pm 0.4) \times 10^{-3}$		2306
Γ_9	$\overline{D}_0^*(2420)^0\ell^+\nu_\ell$, $\overline{D}_0^{*0} \to D^-\pi^+$		$(2.5 \pm 0.5) \times 10^{-3}$		
Γ_{10}	$\overline{D}_2^*(2460)^0\ell^+\nu_\ell$, $\overline{D}_2^{*0} \to D^-\pi^+$		$(1.53 \pm 0.16) \times 10^{-3}$	S=1.0	2065
Γ_{11}	$D^{(*)}$ n $\pi\ell^+\nu_\ell$ (n $\geq$ 1)		$(1.88 \pm 0.25)\%$		
Γ_{12}	$D^{*-}\pi^+\ell^+\nu_\ell$		$(6.0 \pm 0.4) \times 10^{-3}$		2254
Γ_{13}	$\overline{D}_1(2420)^0\ell^+\nu_\ell$, $\overline{D}_1^0 \to D^{*-}\pi^+$		$(3.03 \pm 0.20) \times 10^{-3}$		2084
Γ_{14}	$\overline{D}_1'(2430)^0\ell^+\nu_\ell$, $\overline{D}_1'^0 \to D^{*-}\pi^+$		$(2.7 \pm 0.6) \times 10^{-3}$		
Γ_{15}	$\overline{D}_2^*(2460)^0\ell^+\nu_\ell$, $\overline{D}_2^{*0} \to D^{*-}\pi^+$		$(1.01 \pm 0.24) \times 10^{-3}$	S=2.0	2065
Γ_{16}	$\overline{D}^0\pi^+\pi^-\ell^+\nu_\ell$		$(1.7 \pm 0.4) \times 10^{-3}$		2301
Γ_{17}	$\overline{D}^{*0}\pi^+\pi^-\ell^+\nu_\ell$		$(8 \pm 5) \times 10^{-4}$		2248
Γ_{18}	$D_s^{(*)-}K^+\ell^+\nu_\ell$		$(6.1 \pm 1.0) \times 10^{-4}$		
Γ_{19}	$D_s^-K^+\ell^+\nu_\ell$		$(3.0^{+1.4}_{-1.2}) \times 10^{-4}$		2242
Γ_{20}	$D_s^{*-}K^+\ell^+\nu_\ell$		$(2.9 \pm 1.9) \times 10^{-4}$		2185
Γ_{21}	$\pi^0\ell^+\nu_\ell$		$(7.80 \pm 0.27) \times 10^{-5}$		2638
Γ_{22}	$\pi^0 e^+\nu_e$				2638
Γ_{23}	$\eta\ell^+\nu_\ell$		$(3.9 \pm 0.5) \times 10^{-5}$		2611
Γ_{24}	$\eta'\ell^+\nu_\ell$		$(2.3 \pm 0.8) \times 10^{-5}$		2553
Γ_{25}	$\omega\ell^+\nu_\ell$	[1]	$(1.19 \pm 0.09) \times 10^{-4}$		2582
Γ_{26}	$\omega\mu^+\nu_\mu$				2581
Γ_{27}	$\rho^0\ell^+\nu_\ell$	[1]	$(1.58 \pm 0.11) \times 10^{-4}$		2583
Γ_{28}	$p\overline{p}\ell^+\nu_\ell$		$(5.8^{+2.6}_{-2.3}) \times 10^{-6}$		2467
Γ_{29}	$p\overline{p}\mu^+\nu_\mu$		$< 8.5 \times 10^{-6}$	CL=90%	2446
Γ_{30}	$p\overline{p}e^+\nu_e$		$(8.2^{+4.0}_{-3.3}) \times 10^{-6}$		2467
Γ_{31}	$e^+\nu_e$		$< 9.8 \times 10^{-7}$	CL=90%	2640
Γ_{32}	$\mu^+\nu_\mu$		2.90E-07 to 1.07E-06	CL=90%	2639
Γ_{33}	$\tau^+\nu_\tau$		$(1.09 \pm 0.24) \times 10^{-4}$	S=1.2	2341
Γ_{34}	$\ell^+\nu_\ell\gamma$		$< 3.0 \times 10^{-6}$	CL=90%	2640
Γ_{35}	$e^+\nu_e\gamma$		$< 4.3 \times 10^{-6}$	CL=90%	2640
Γ_{36}	$\mu^+\nu_\mu\gamma$		$< 3.4 \times 10^{-6}$	CL=90%	2639
Γ_{37}	$\mu^+\mu^-\mu^+\nu_\mu$		$< 1.6 \times 10^{-8}$	CL=95%	2634



For inclusive decay many channels open and no lattice results — only analytical methods exist

But recently new ideas!

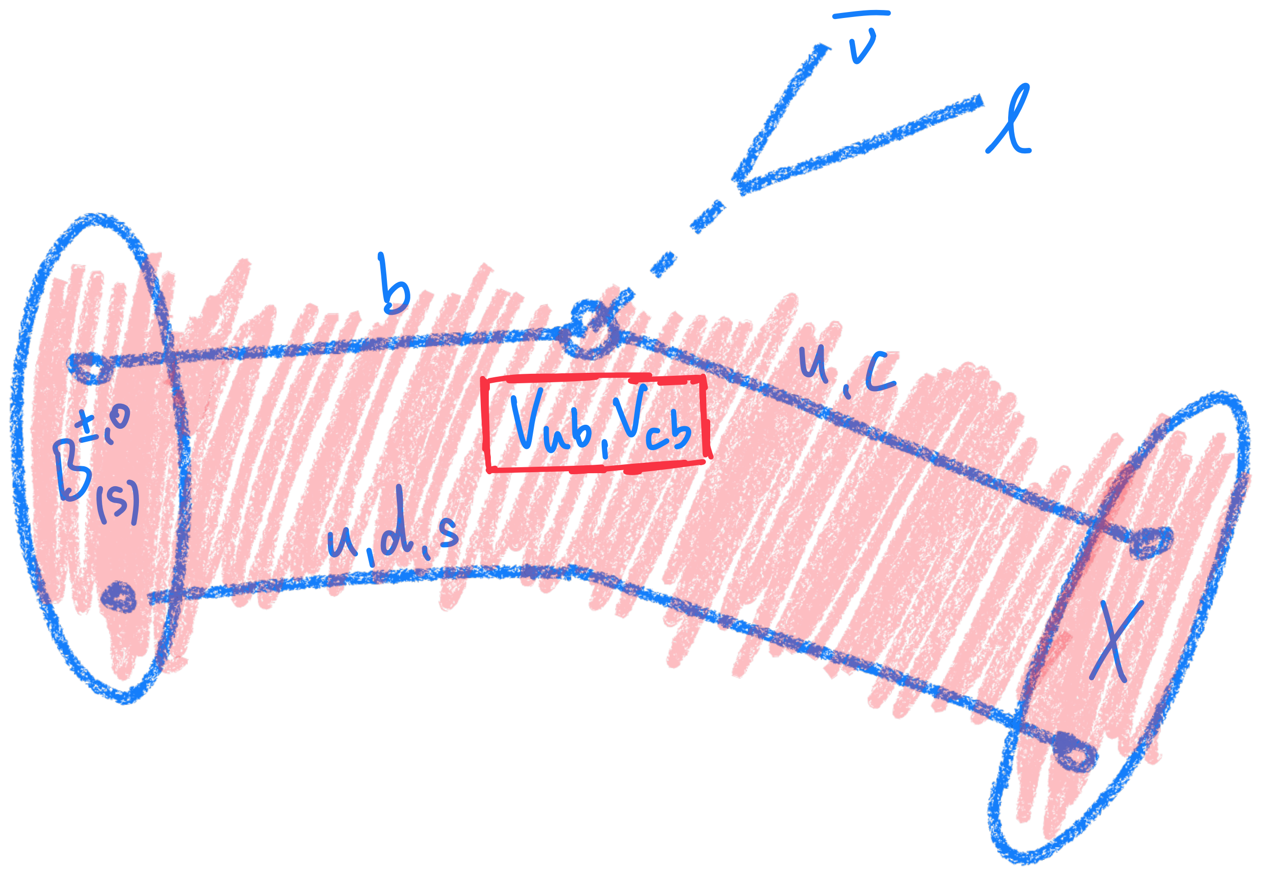
Hansen et al. (2017) PRD 96 094513 (2017)
Hashimoto PTEP 53-56 (2017)
Bailas et al. PTEP 43-50 (2020)
Gambino and Hashimoto PRL 125 32001 (2020)
Gambino et al. JHEP 07 (2022) 083
Barone et al. JHEP 07 (2023) 145
Barone et al. arXiv:2504.03358
De Santis et al. arXiv:2504.06063
De Santis et al. arXiv:2504.06064

Inclusive SL decay in the SM

$$\frac{d\Gamma}{dq^2 dq^0 dE_l} = \frac{G_F^2 |V_{qQ}|^2}{8\pi} \boxed{L^{\mu\nu}} \boxed{W_{\mu\nu}}$$

hadronic tensor

leptonic tensor



We consider the case $B_s \to X_c \ell \nu$**:**

$$W^{\mu\nu}(p_{B_s}, q) = \frac{1}{2E_{B_s}} \sum_{X_c} (2\pi)^3 \delta^{(4)}(p_{B_s} - q - p_{X_c}) \langle B_s(\mathbf{p}_{B_s}) | (\tilde{J}^\mu(q^2))^\dagger | X_c(p_{X_c}) \rangle \langle X_c(p_{X_c}) | \tilde{J}^\nu(q^2) | B_s(\mathbf{p}_{B_s}) \rangle$$

from now on B_s at rest ($\mathbf{p}_{B_s} = \mathbf{0}$)

Inclusive SL decay

kinematics:

$$\omega_{\min} = \sqrt{M_{D_s}^2 + \mathbf{q}^2}$$

$$\omega_{\max} = M_{B_s} - \sqrt{\mathbf{q}^2}$$

$$\mathbf{q}^2_{\max} = \frac{(M_{B_s}^2 - M_{D_s}^2)^2}{4M_{B_s}^2}$$

$$\frac{d\Gamma}{dq^2 dq^0 dE_l} = \frac{G_F^2 |V_{cb}|^2}{8\pi} L^{\mu\nu} W_{\mu\nu}$$

Integrate phase space:

$$\Gamma(D_s \to Xl\nu) = \frac{G_F^2 |V_{cs}|^2}{24\pi^3} \int_0^{\mathbf{q}^2_{\max}} d\mathbf{q}^2 \sqrt{\mathbf{q}^2}\, \boxed{\bar{X}(\mathbf{q}^2)}$$

where

$$\boxed{\bar{X}(\mathbf{q}^2) = \int_{\omega_{\min}}^{\infty} d\omega\, W_{\mu\nu}(\omega, \mathbf{q}) K^{\mu\nu}(\omega, \mathbf{q})}$$

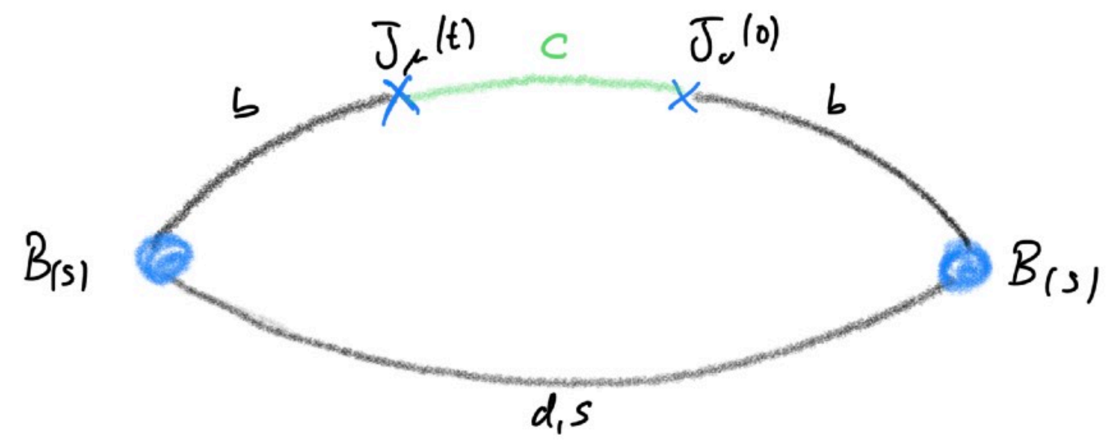
- integration over lepton energy $E_l \to K^{\mu\nu}$
- ω is energy of intermediate state X_c
- $\mathbf{q}$ is three-momentum transfer

Inclusive SL decay

$$\bar{X}(\mathbf{q}) = \int_{\omega_0}^{\infty} d\omega\, W_{\mu\nu}(\omega, \mathbf{q})\, K^{\mu\nu}(\omega, \mathbf{q})$$

$$W_{\mu\nu}(\omega, \mathbf{q}) = \frac{1}{2M_{B_s}} \langle B_s(0) \,|\, \tilde{J}_\mu^\dagger(q) \tilde{J}_\nu(q) \,|\, B_s(0) \rangle$$

$$C_{\mu\nu}(t, \mathbf{q}) = \frac{C_{4\text{pt}}^{\mu\nu}(t_2, t_1)}{C_{2\text{pt}}(t_2) C_{2\text{pt}}(t_1)} \to \sum_{\mathbf{x}} \frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{2M_{B_s}} \langle B_s \,|\, J_\mu^\dagger(\mathbf{x}, t)\, J_\nu(\mathbf{0},0) \,|\, B_s \rangle \qquad (t = t_2 - t_1 \geq 0)$$



$$= \int_{\omega_{\min}}^{\infty} d\omega \frac{1}{2M_{B_s}} \langle B_s \,|\, (\tilde{J}_\mu(\mathbf{q},0))^\dagger \delta(\hat{H} - \omega) \tilde{J}_\nu(\mathbf{q},0) \,|\, B_s \rangle\, e^{-t\omega}$$

$$\boxed{C_{\mu\nu}(t, \mathbf{q}) = \int_{\omega_{\min}}^{\infty} d\omega\, W_{\mu\nu}(\omega, \mathbf{q})\, e^{-\omega t}}$$

$\omega_{\min}$ is mass of lightest final state

Euclidean 4pt function is Laplace transform of hadronic tensor

$\bar{X}(\mathbf{q})$ reconstruction

$$\bar{X}(\mathbf{q}) = \int_{\omega_0}^{\infty} d\omega\, W_{\mu\nu}(\omega, \mathbf{q})\, \boxed{K^{\mu\nu}(\mathbf{q}, \omega)} \quad \longleftrightarrow \quad C_{\mu\nu}(t, \mathbf{q}) = \int_{\omega_0}^{\infty} d\omega\, W_{\mu\nu}(\omega, \mathbf{q})\, \boxed{e^{-\omega t}}$$

We can expand the kernel K (analytically known) in powers of $e^{-a\omega}$:

$$\begin{aligned}
\bar{X}(\mathbf{q}) &\approx c_{\mu\nu,0}(\mathbf{q}) \int_{\omega_0}^{\infty} d\omega W_{\mu\nu}(\omega, \mathbf{q}) &&+ c_{\mu\nu,1}(\mathbf{q}) \int_{\omega_0}^{\infty} d\omega W_{\mu\nu}(\omega, \mathbf{q})\, e^{-a\omega} &&+ c_{\mu\nu,2}(\mathbf{q}) \int_{\omega_0}^{\infty} d\omega W_{\mu\nu}(\omega, \mathbf{q})\, e^{-2a\omega} &&+ \ldots \\
&= c_{\mu\nu,0}(\mathbf{q})\;\; C_{\mu\nu}(0, \mathbf{q}) &&+ c_{\mu\nu,1}(\mathbf{q})\;\; C_{\mu\nu}(a, \mathbf{q}) &&+ c_{\mu\nu,2}(\mathbf{q})\;\; C_{\mu\nu}(2a, \mathbf{q}) &&+ \ldots
\end{aligned}$$

- fully determined linear system
- $\bar{X}$ fully determined once we know coefficients $c_{\mu\nu,k}(\mathrm{q})$

$$\begin{aligned}
\bar{X}(\mathbf{q}) &= \sum_k c_{\mu\nu,k}(\mathbf{q}) \int d\omega W_{\mu\nu}(\omega, \mathbf{q})\, e^{-\omega k} \\
&= \sum_k c_{\mu\nu,k}(\mathbf{q})\, C_{\mu\nu}(ak, \mathbf{q})
\end{aligned}$$

$\bar{X}(\mathbf{q})$ reconstruction

$$\bar{X}(\mathbf{q}) = \sum_k c_{\mu\nu,k}(\mathbf{q}) \int d\omega W_{\mu\nu}(\omega, \mathbf{q})\, e^{-\omega k}$$
$$= \sum_k c_{\mu\nu,k}(\mathbf{q})\, C_{\mu\nu}(ak, \mathbf{q})$$

In theory:

- coefficients $c_{\mu\nu,k}$ known analytically
- $C_{\mu\nu}(t)$ can be computed on the lattice
- in principle well-defined solution for $\bar{X}(\mathrm{q})$

In practice:

- signal-to-noise deteriorates with t
- in practice not sufficient to extract meaningful signal

Similar to the case of exclusive decay we need a regulator for the badly determined higher-order (larger t) terms

Kernel approximation

$$\bar{X}(\mathbf{q}) \approx c_{\mu\nu,0}(\mathbf{q})\ \ C_{\mu\nu}(0,\mathbf{q}) + c_{\mu\nu,1}(\mathbf{q})\ \ C_{\mu\nu}(a,\mathbf{q}) + c_{\mu\nu,2}(\mathbf{q})\ \ C_{\mu\nu}(2a,\mathbf{q}) + \dots$$

instead of in $(e^{-a\omega})^n$ we now expand in shifted Chebyshev polynomials

Barata, Fredenhagen, Commun.Math.Phys. 138 (1991) 507-520, Bailas et al. PTEP 43-50 (2020), Gambino and Hashimoto PRL 125 32001 (2020)

$$\tilde{T}_k(\omega)\ :\ [\omega_0, \infty] \to [-1,1]$$

$$\tilde{T}_k(\omega) = \sum_{j=0}^{k} \tilde{t}_j^{(k)}\, e^{-ja\omega}$$

$$K_{\mu\nu}(\omega,\mathbf{q}) \approx \frac{\tilde{c}_{\mu\nu,0}}{2} + \sum_{k=1}^{N} \tilde{c}_{\mu\nu,k}\tilde{T}_k(\omega)$$

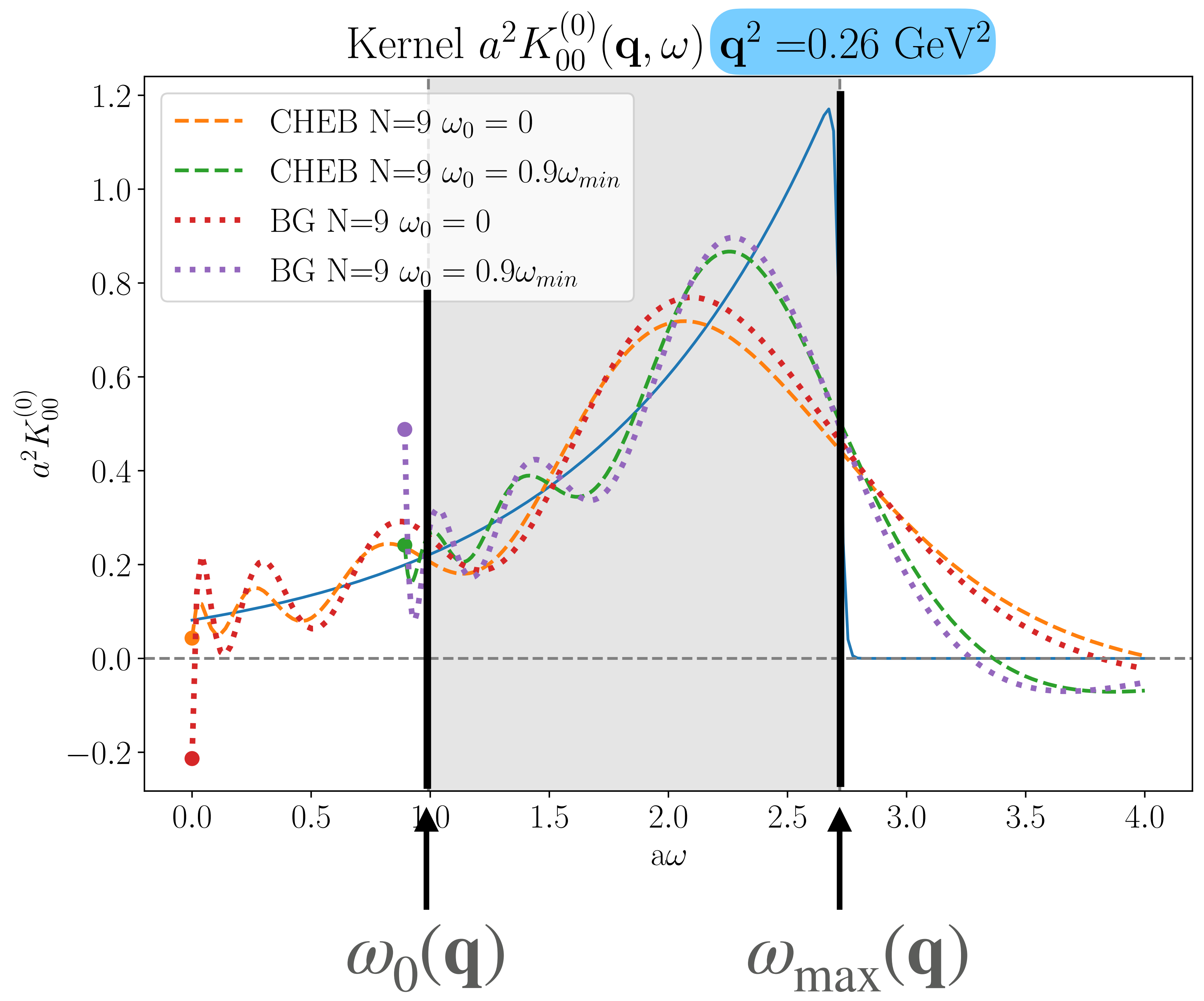
$$\tilde{c}_{\mu\nu,k} = \int_{\omega_0}^{\infty} d\omega\, K_{\mu\nu}(\omega,\mathbf{q})\tilde{T}_k(\omega)\tilde{\Omega}(\omega)$$

Kernel approximation

$$K_{\mu\nu}(\omega, \mathbf{q}; t_0) = e^{2\omega t_0} k_{\mu\nu}(\omega, \mathbf{q},)\theta_\sigma(\omega_{\max} - \omega)$$

$$\tilde{c}_k = \langle K, \tilde{T}_k \rangle = \int_{\omega_0}^{\infty} d\omega\, K_{\mu\nu}(\omega, \mathbf{q}) \tilde{T}_k(\omega) \tilde{\Omega}(\omega)$$

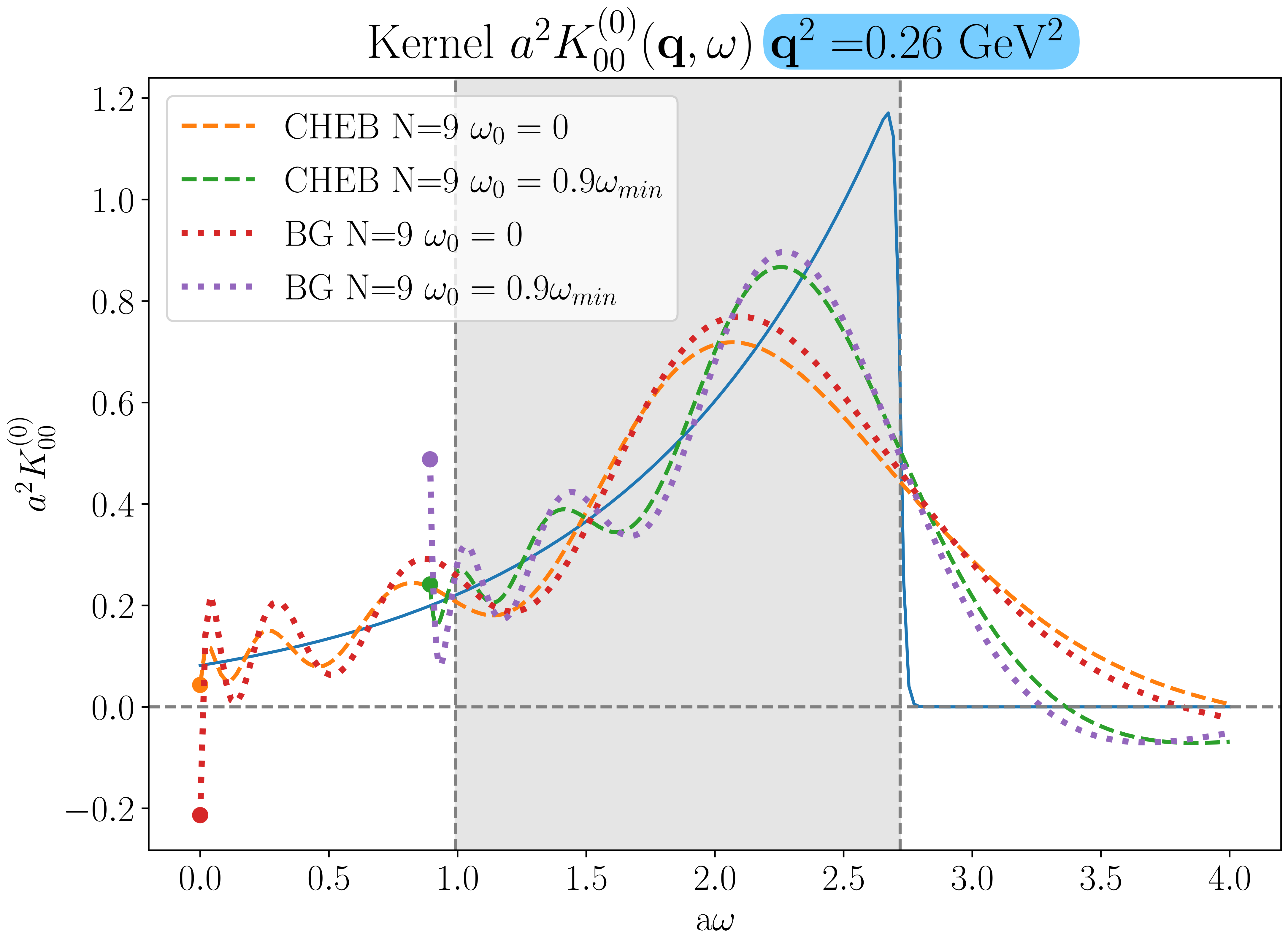
e.g. $K^{(0)}_{\sigma,00}(\mathbf{q}, \omega; t_0) = e^{2\omega t_0}\, \mathbf{q}^2\, \theta_\sigma(\omega_{\max} - \omega)$

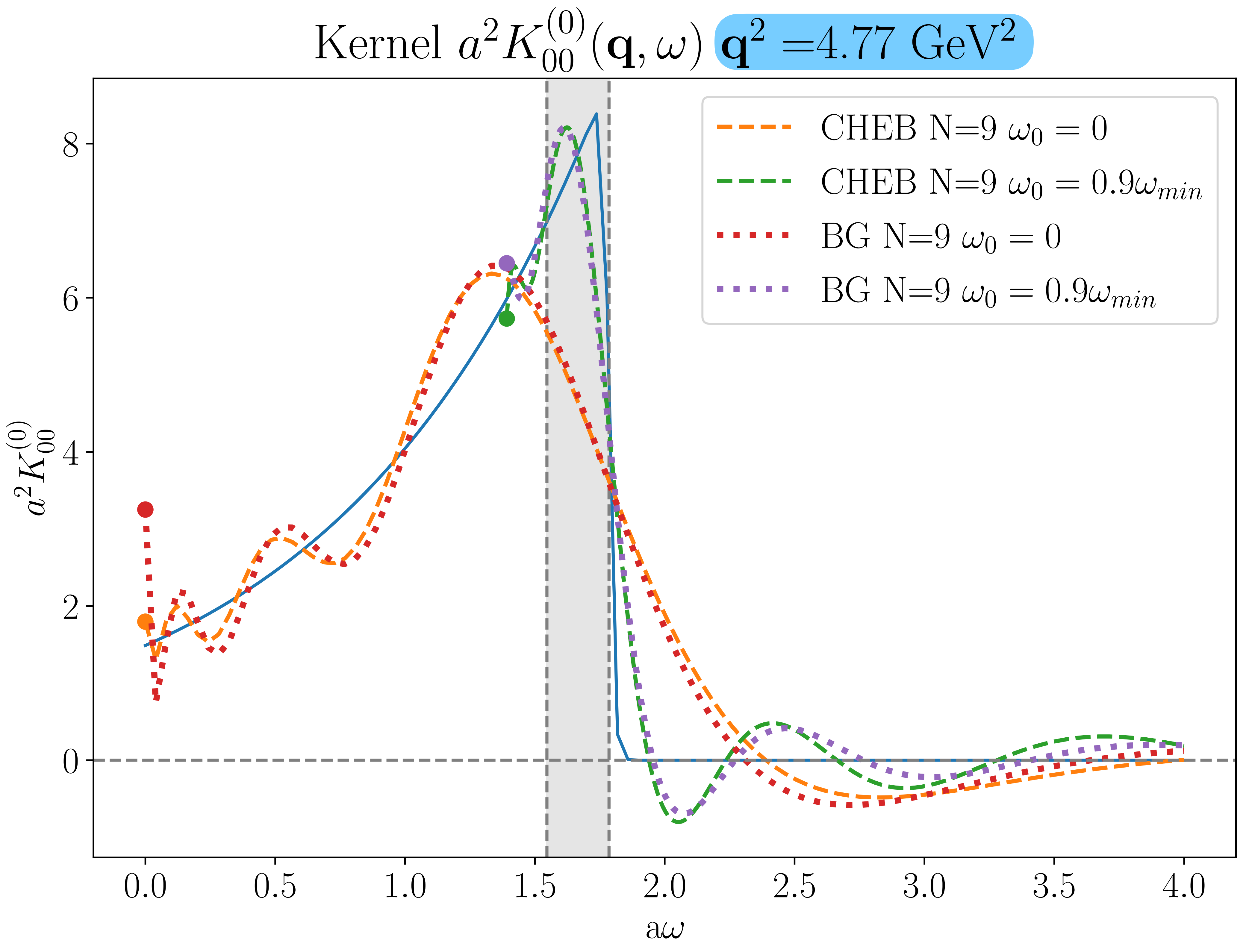


- this analysis stage independent of data
- smearing σ
- order of approximation $N \leftrightarrow C_{\mu\nu}(t)$
- we can play with ω_0

Kernel approximation

$$K_{\mu\nu}(\mathbf{q}, \omega, t_0) = e^{2\omega t_0} k_{\mu\nu}(\mathbf{q}, \omega)\theta_\sigma(\omega_{\max} - \omega)$$





$\bar{X}(\mathbf{q})$ reconstruction

$$\bar{X}(\mathbf{q}) = \sum_k \tilde{c}_{\mu\nu,k}(\mathbf{q}) \int d\omega W_{\mu\nu}(\omega, \mathbf{q})\, \tilde{T}_k(\omega)$$
$$= \sum_{k,j} \tilde{c}_{\mu\nu,k}(\mathbf{q})\, \langle \tilde{T}_k \rangle_{\mu\nu}$$

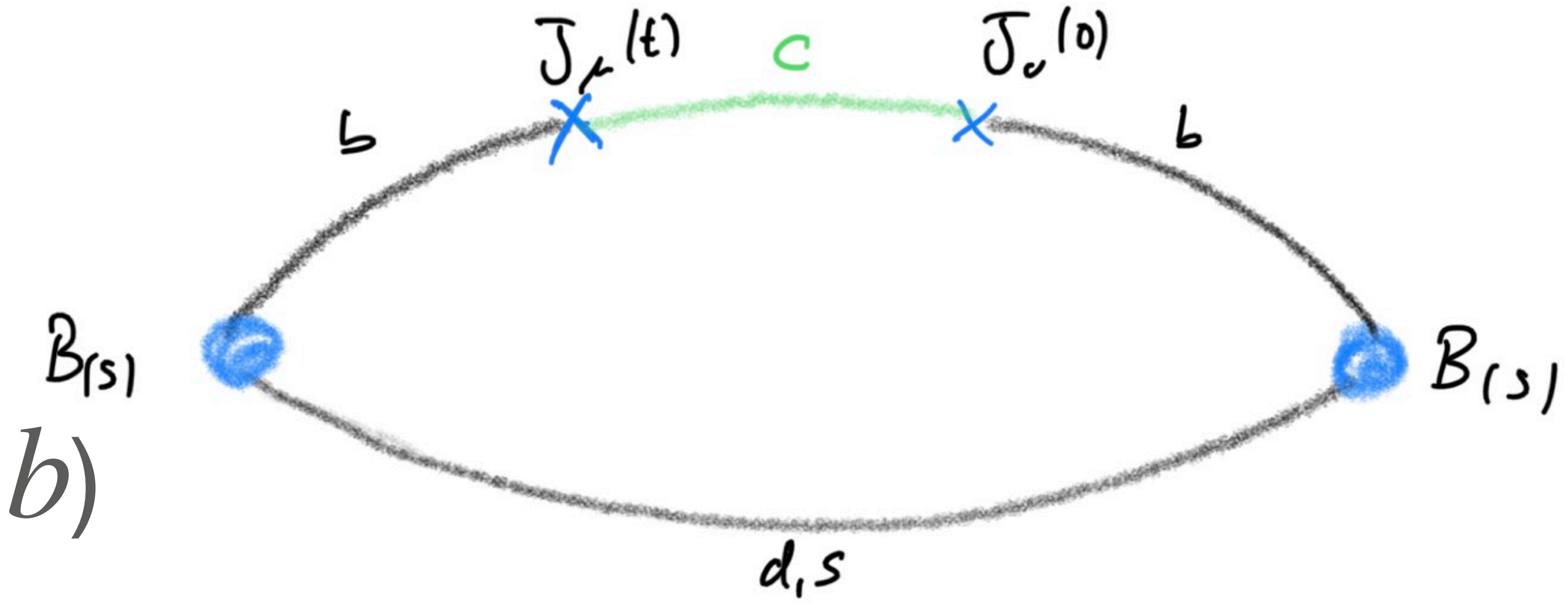
$$\langle \tilde{T}_k \rangle_{\mu\nu} = \frac{\sum_{j=0}^{k} \tilde{t}_j^{(k)} C_{\mu\nu}(j + 2t_0)}{C_{\mu\nu}(2t_0)}$$

- $C_{\mu\nu}(t)$ from lattice
- $C_{\mu\nu}(t)$ monotonously decreasing with t
- **we use $|\langle \tilde{T}_k \rangle_{\mu\nu}| \le 1$ as uniform Bayesian prior (i.e. regulator)**
- for sufficiently smooth kernel the $\tilde{c}_{\mu\nu,k}$ turn out to be nicely behaved

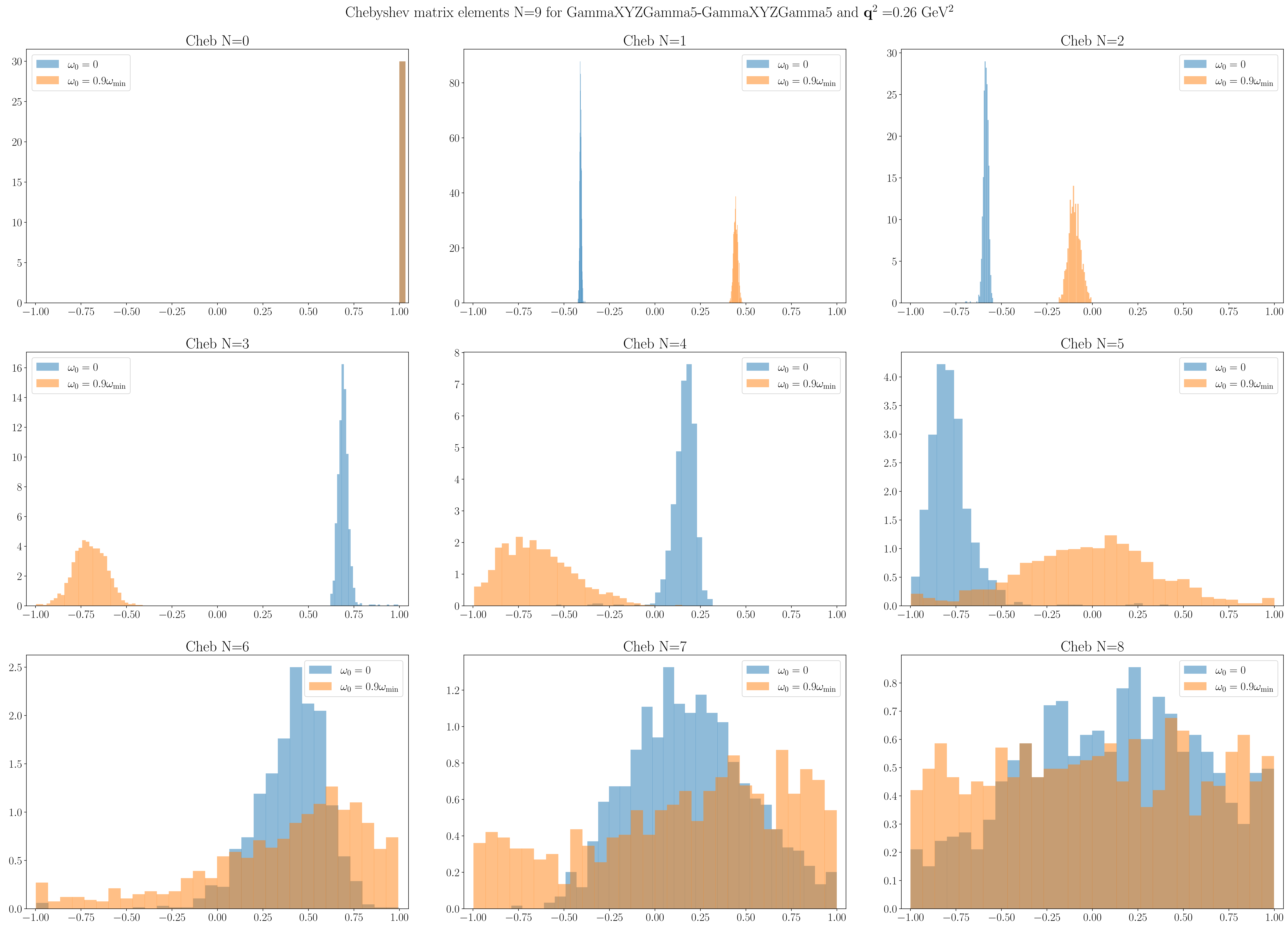
  $\rightarrow$ suppression of higher-order terms

Exploratory study

- $B_s \to X_c \ell \nu$
- lattice study on $24^3 \times 64$ RBC/UKQCD DWF ensemble ($M_\pi^{\rm sea} \approx 330\,\mathrm{MeV}$)
- physical m_s- and m_b-quark masses (RHQ action for b) near-physical m_c (domain-wall)
- implemented in Grid/Hadrons
- run on DiRAC Extreme-scaling service Tursa (A100-40 nodes)
- 120 gauge configs, 8 Z_2 noise-source planes



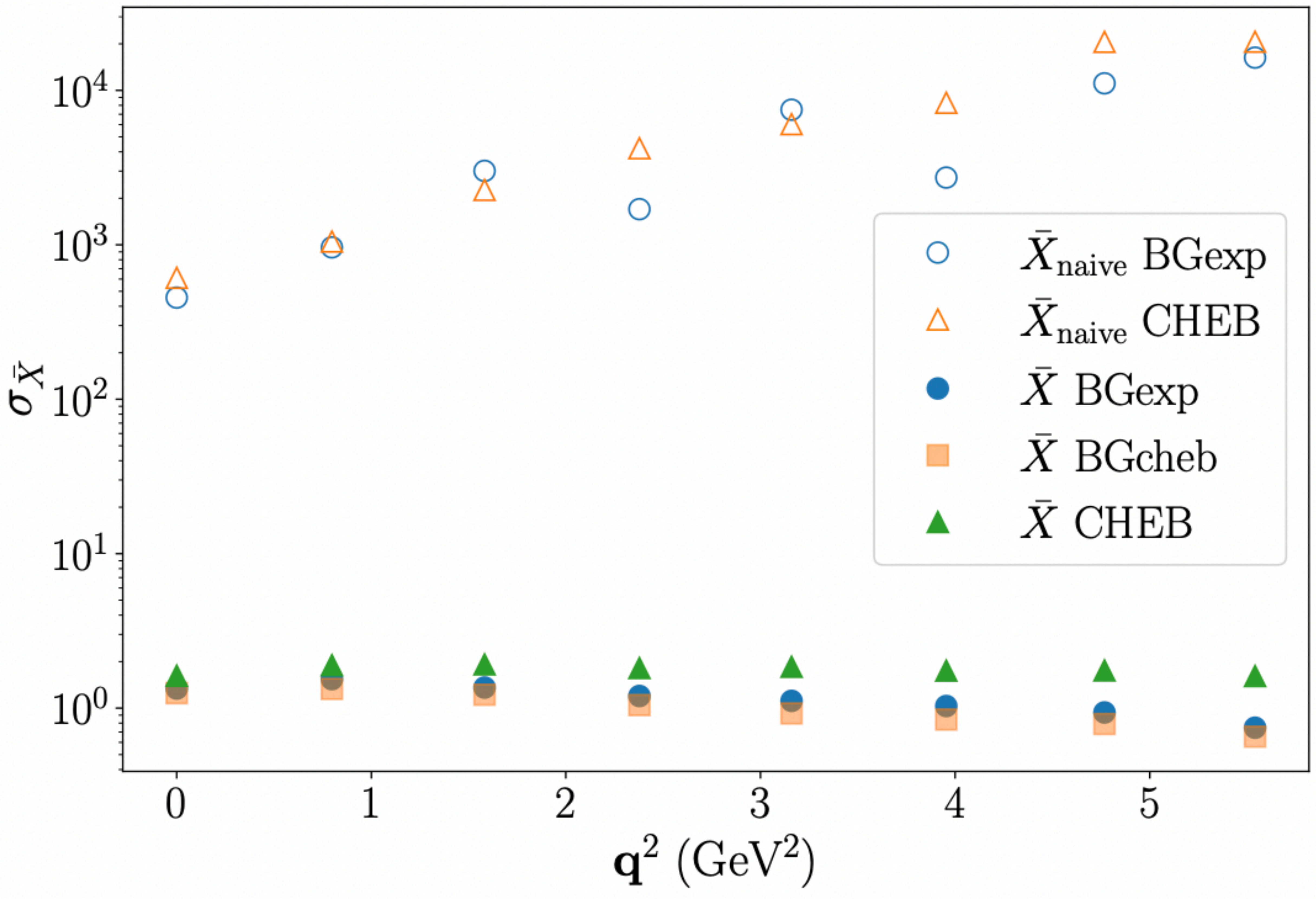
Results for $\langle \tilde{T}_k \rangle_{\mu\nu}$



$$\bar{X}(\mathbf{q}) = \sum_k \tilde{c}_{\mu\nu,k}(\mathbf{q}) \int d\omega W_{\mu\nu}(\omega,\mathbf{q})\,\tilde{T}_k(\omega)$$
$$= \sum_{k,j} \tilde{c}_{\mu\nu,k}(\mathbf{q})\,\langle \tilde{T}_k \rangle_{\mu\nu}$$

- $\omega_0 = 0$ and $\omega_0 = 0.9\,\omega_{\min}$
- successful determination of $\langle \tilde{T}_k \rangle_{\mu\nu}$
- higher orders affected by noise - regulator kicks

Impact of regulator

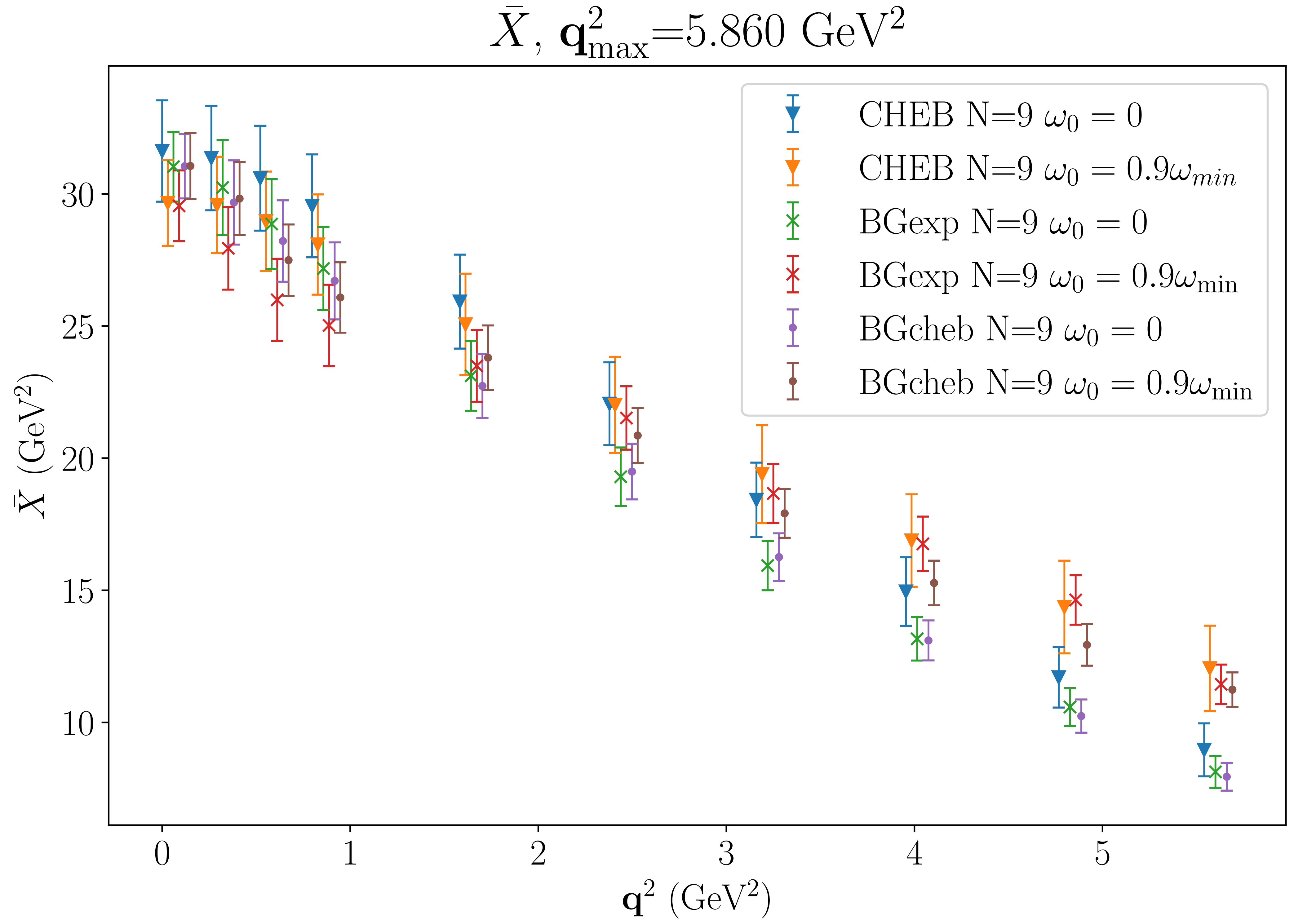


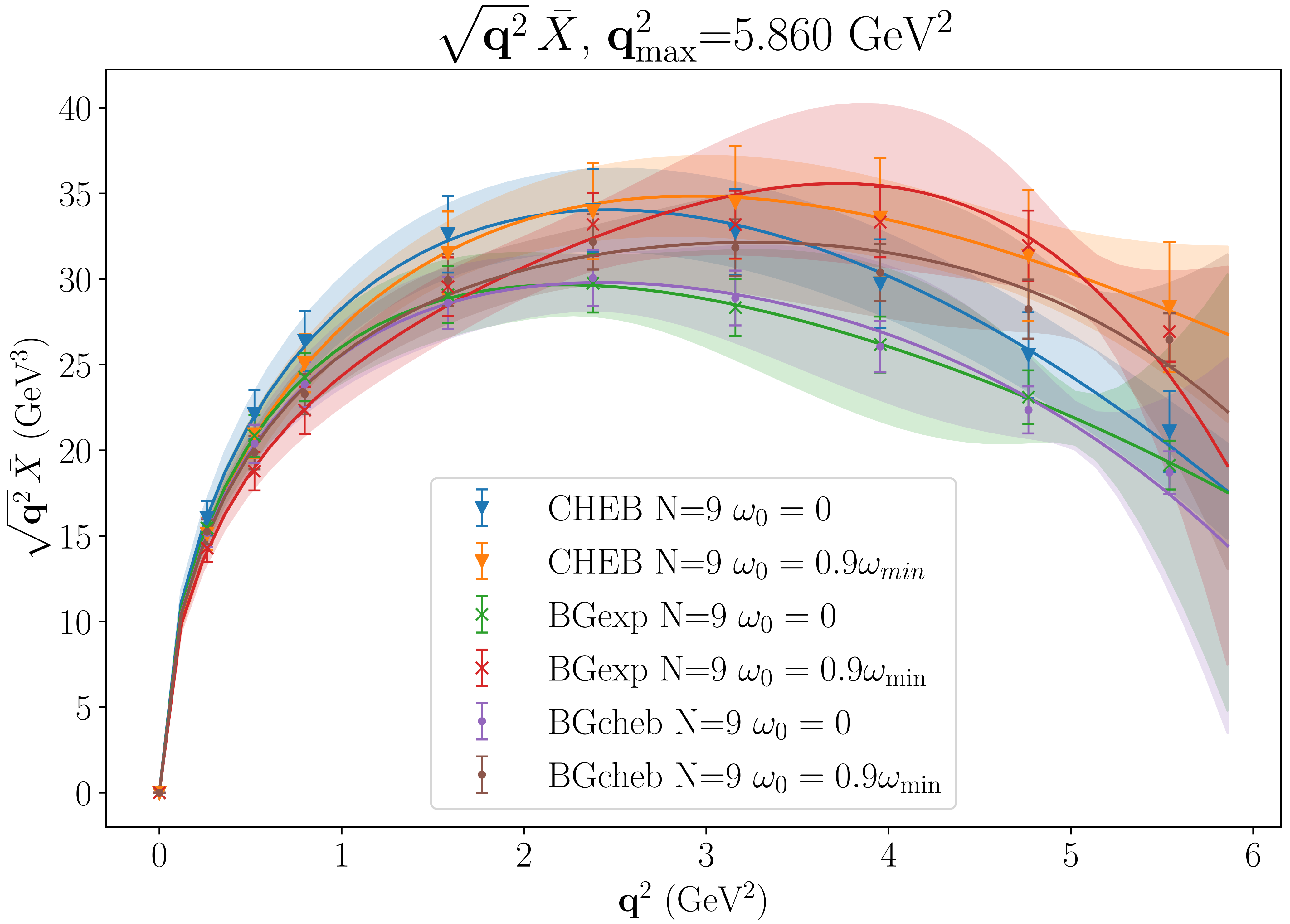
Noise reduction due to regulator term absolutely essential!

"BGexp" is Backus-Gilbert-inspired Hansen-Lupo-Tantalo approach

Hansen et al. (2017) PRD 96 094513 (2017)
De Santis et al. arXiv:2504.06063
De Santis et al. arXiv:2504.06064

Results for $\bar{X}(\mathrm{q})$

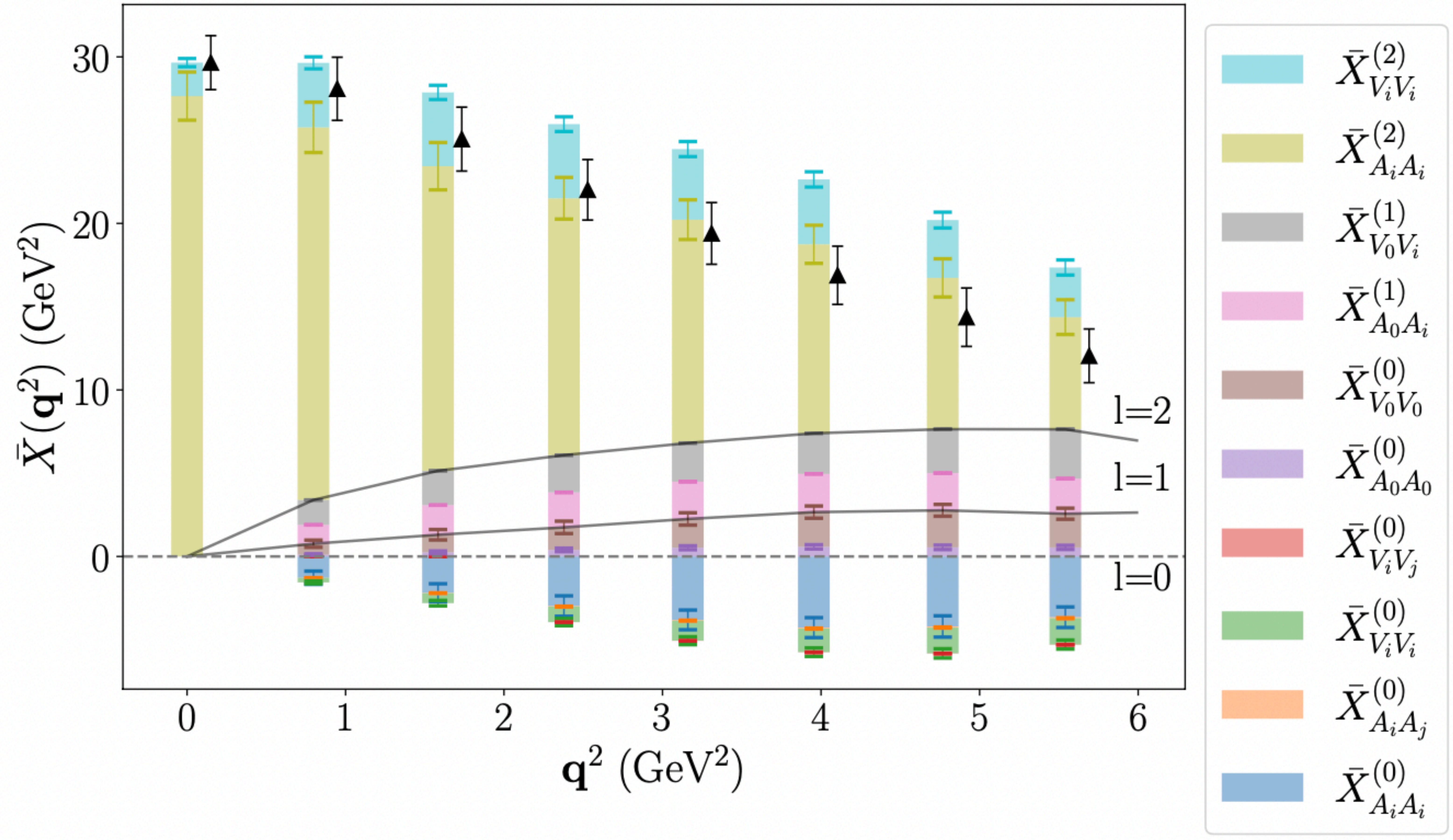




variations of analysis techniques largely consistent — tension at larger $\mathbf{q}^2$ visible

Integral of $\sqrt{\mathbf{q}^2}\,\bar{X}(\mathbf{q}^2)$ proportional to Γ;

Contributions from various channels



Approach provides for nice laboratory to understand and probe contributions to inclusive decay from various sources

Ground-state limit

What is the ground-state contribution to inclusive decay?

We restrict the analysis to the ground-state $B_s \to D_s$ decay:

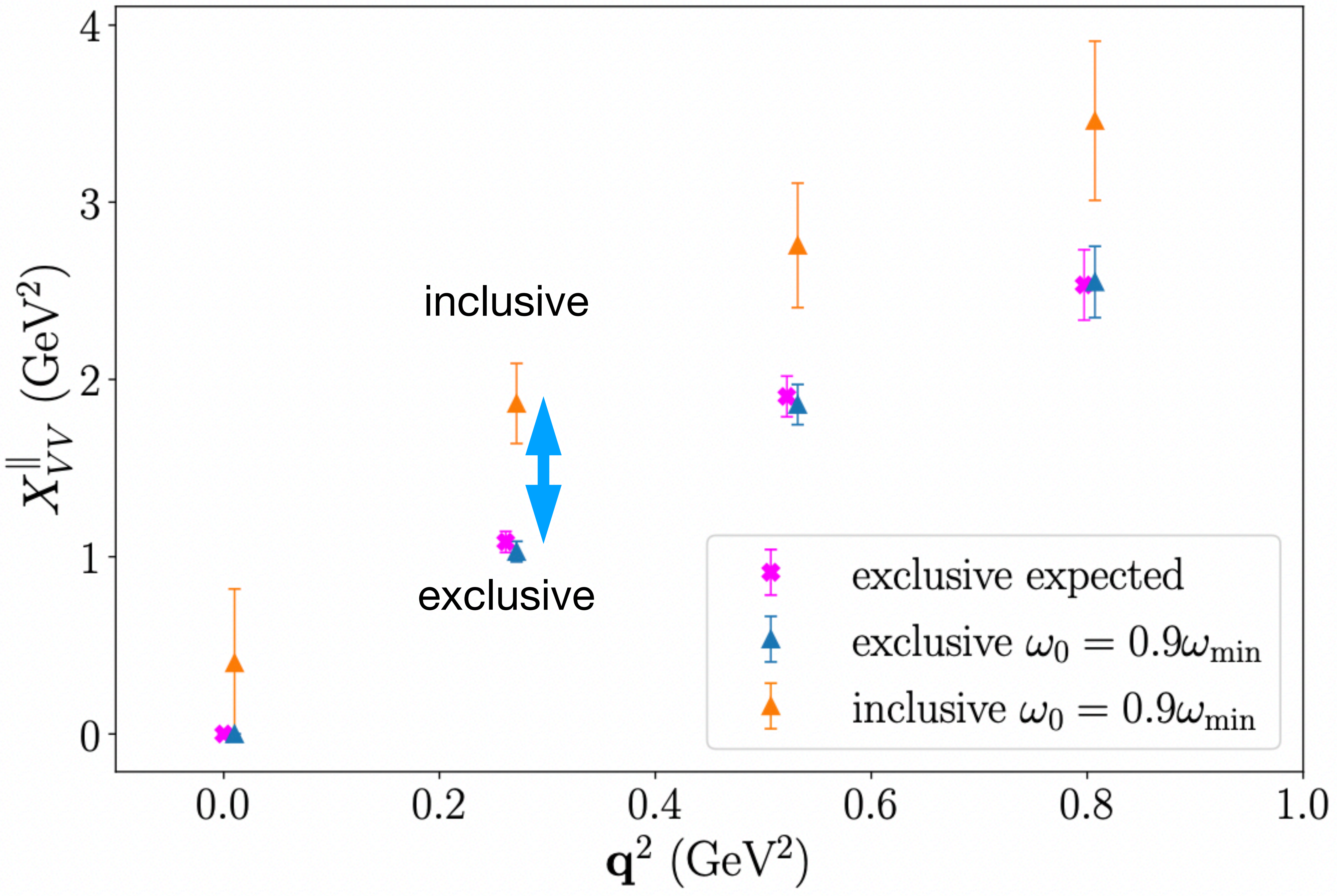
$$W_{\mu\nu} \to \delta(\omega - E_{D_s}) \frac{1}{4M_{B_s}E_{D_s}} \langle B_s | J_\mu^\dagger | D_s \rangle \langle D_s | J_\nu | B_s \rangle$$

The corresponding data is generated on the lattice (analysis of $B_s \to D_s$ 3pt correlators):

$$\langle D_s | V_\mu | B_s \rangle = f_+(q^2)(p_{B_s} + p_{D_s})_\mu + f_-(q^2)(p_{B_s} - p_{D_s})_\mu$$

$$\bar{X}_{VV}^{||} \to \frac{M_{B_s}}{E_{D_s}} \mathbf{q}^2 | f_+(q^2) |^2$$

Ground-state limit



- Results for exclusive channel agree for both ways of data analysis (standard 3pt vs. Chebychev)
- clear distinction between ground-state and full inclusive determination

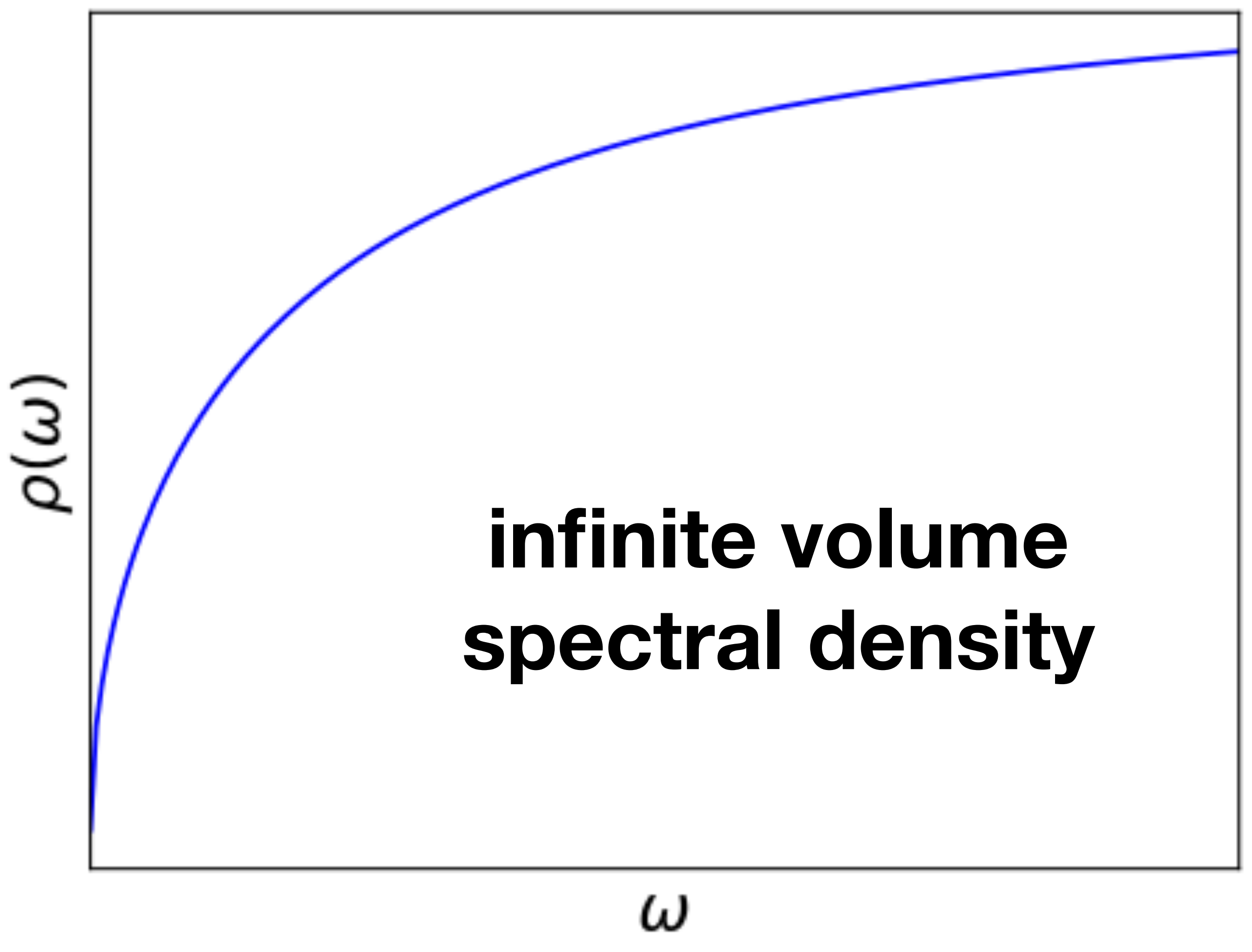
Systematics — finite volume

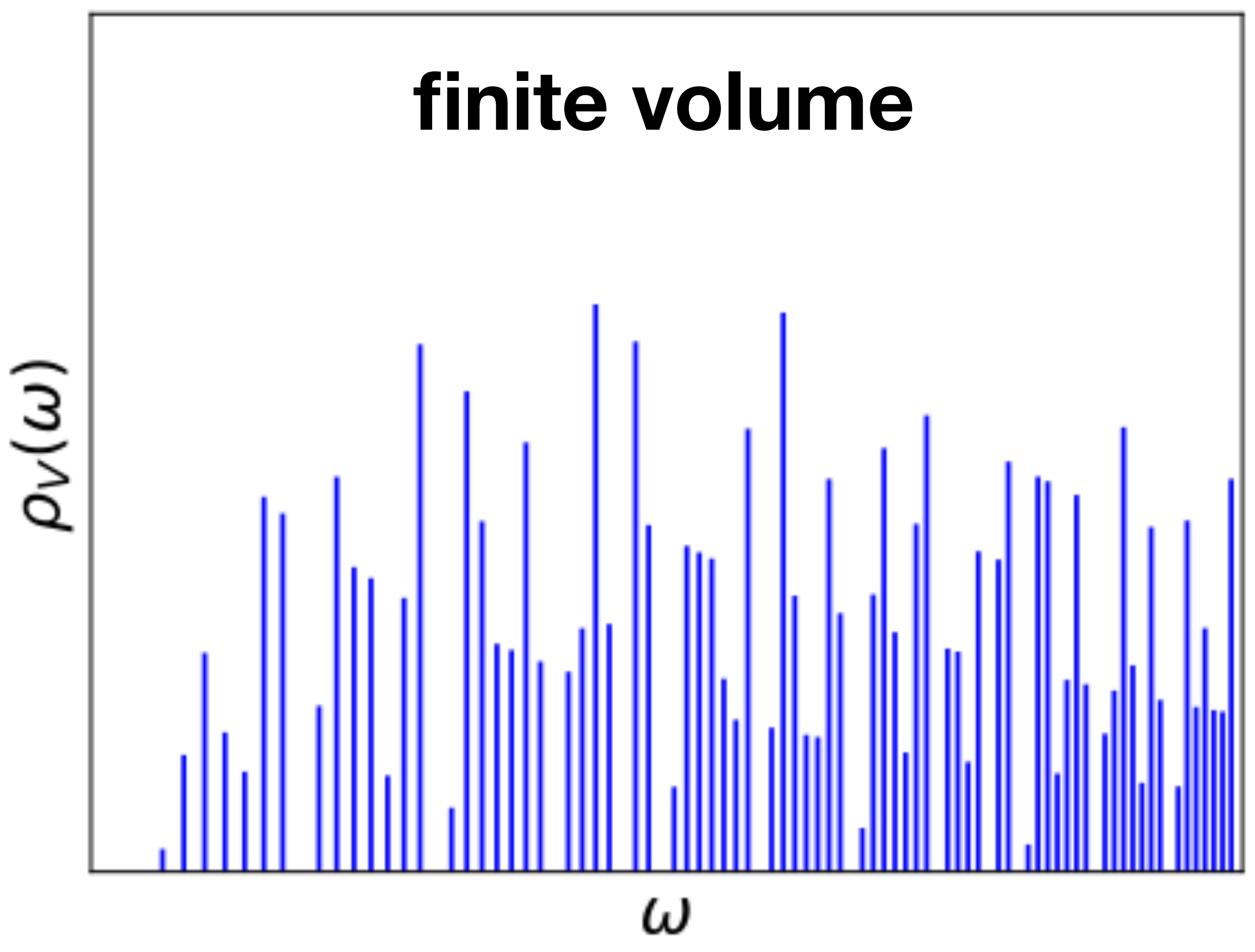
[Kellermann et al. arXiv:2504.03358]

- Order of limits: $\lim_{V\to\infty}\lim_{\sigma\to 0}$
- in practice lattice simulations in finite volume
- need to find ways for estimating effects reliably
- Here: model finite-size effects with spectral density of two non-interacting particles

$$\rho(\omega) = \frac{1}{2\pi}\int_0^\infty dq \frac{q^2}{4(q^2+M^2)}\delta(\omega - 2\sqrt{q^2+M^2})$$

$$\rho_V(\omega) = \frac{\pi}{V}\sum_{\mathbf{q}} \frac{\mathbf{q}^2}{4(\mathbf{q}^2+M^2)}\delta\left(\omega - 2\sqrt{\mathbf{q}^2+M^2}\right)$$

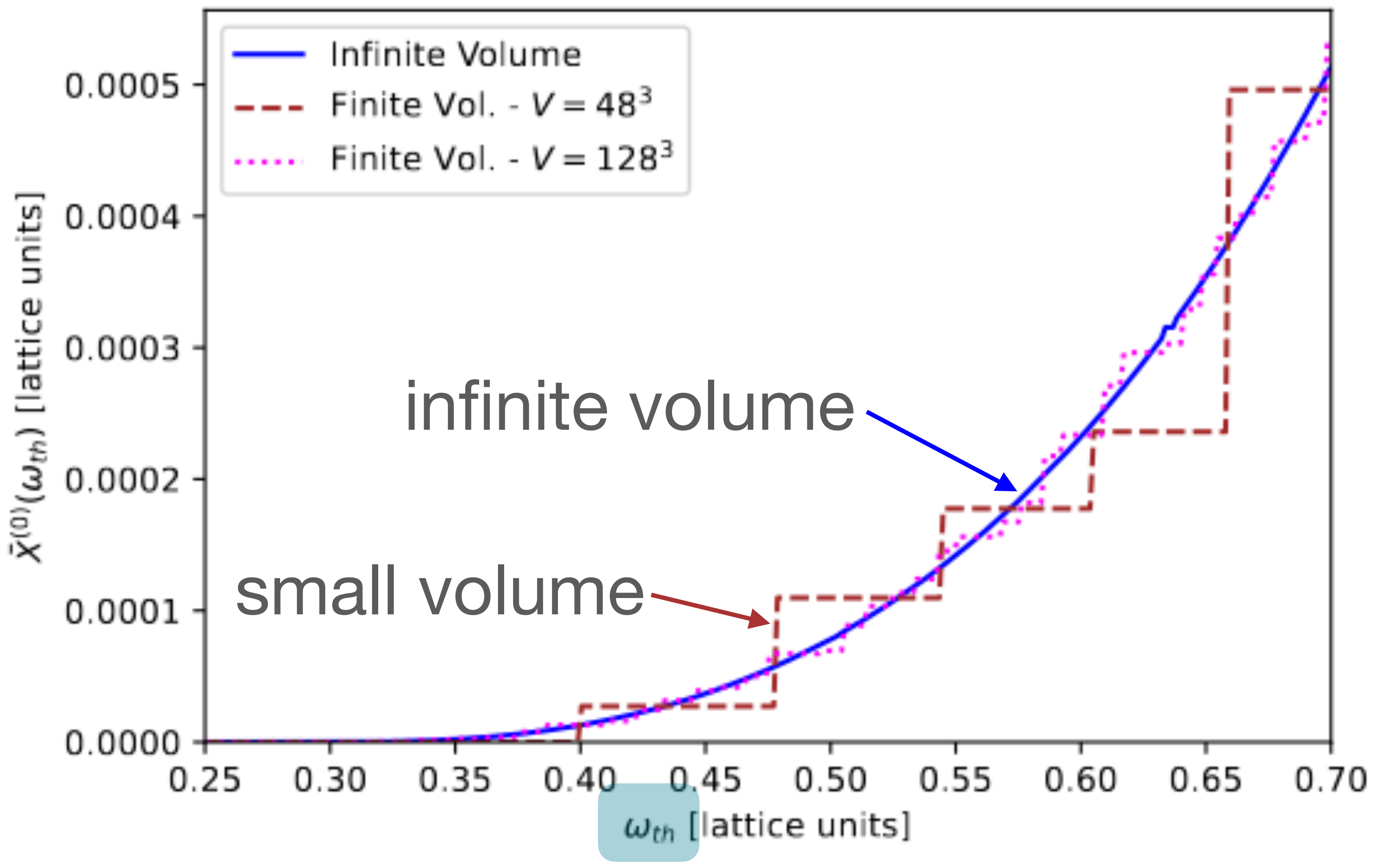


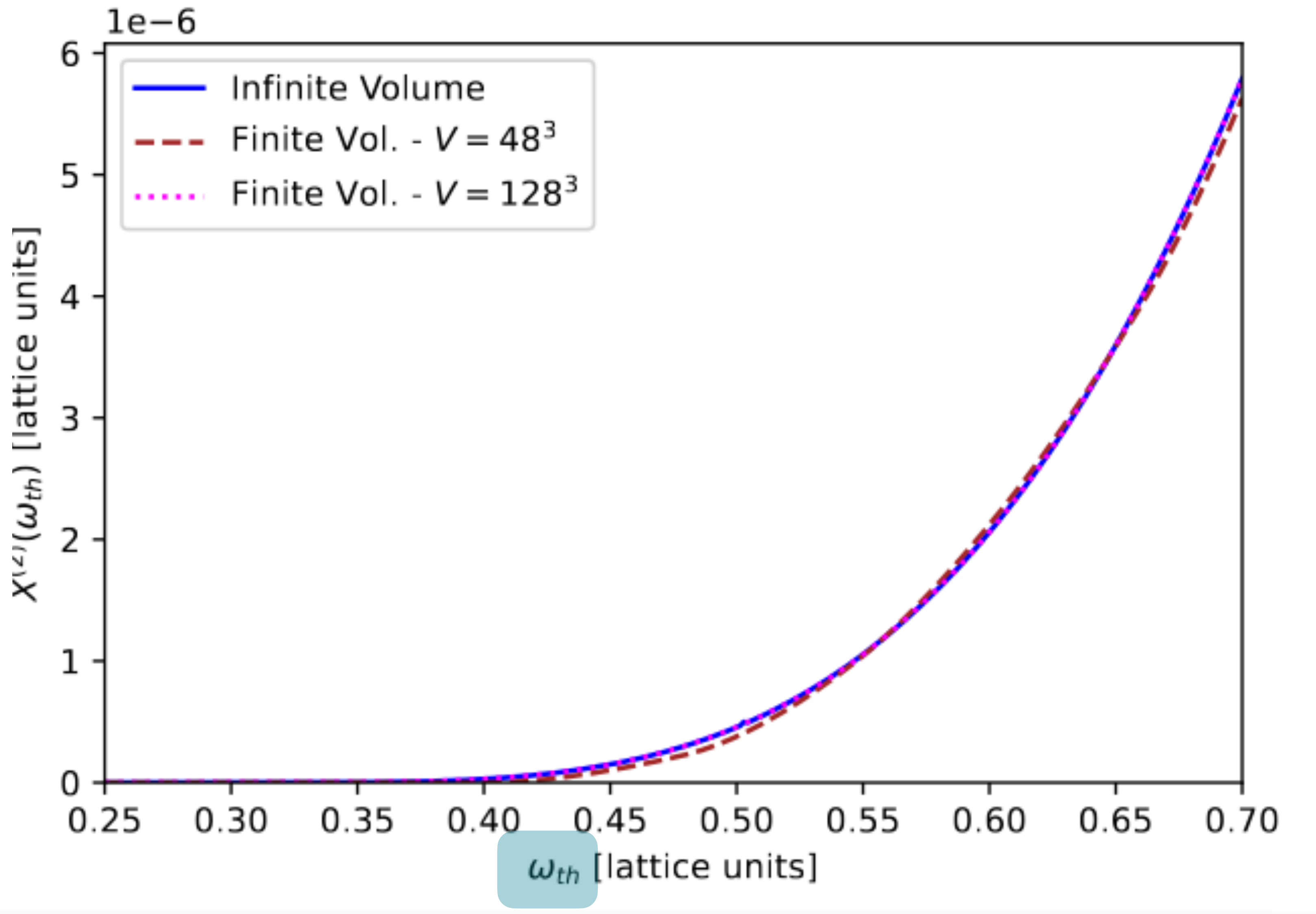


Systematics — finite volume

- Here: compute inclusive rate for two free particles in finite and infinite volume
- vary upper threshold ω_{th}

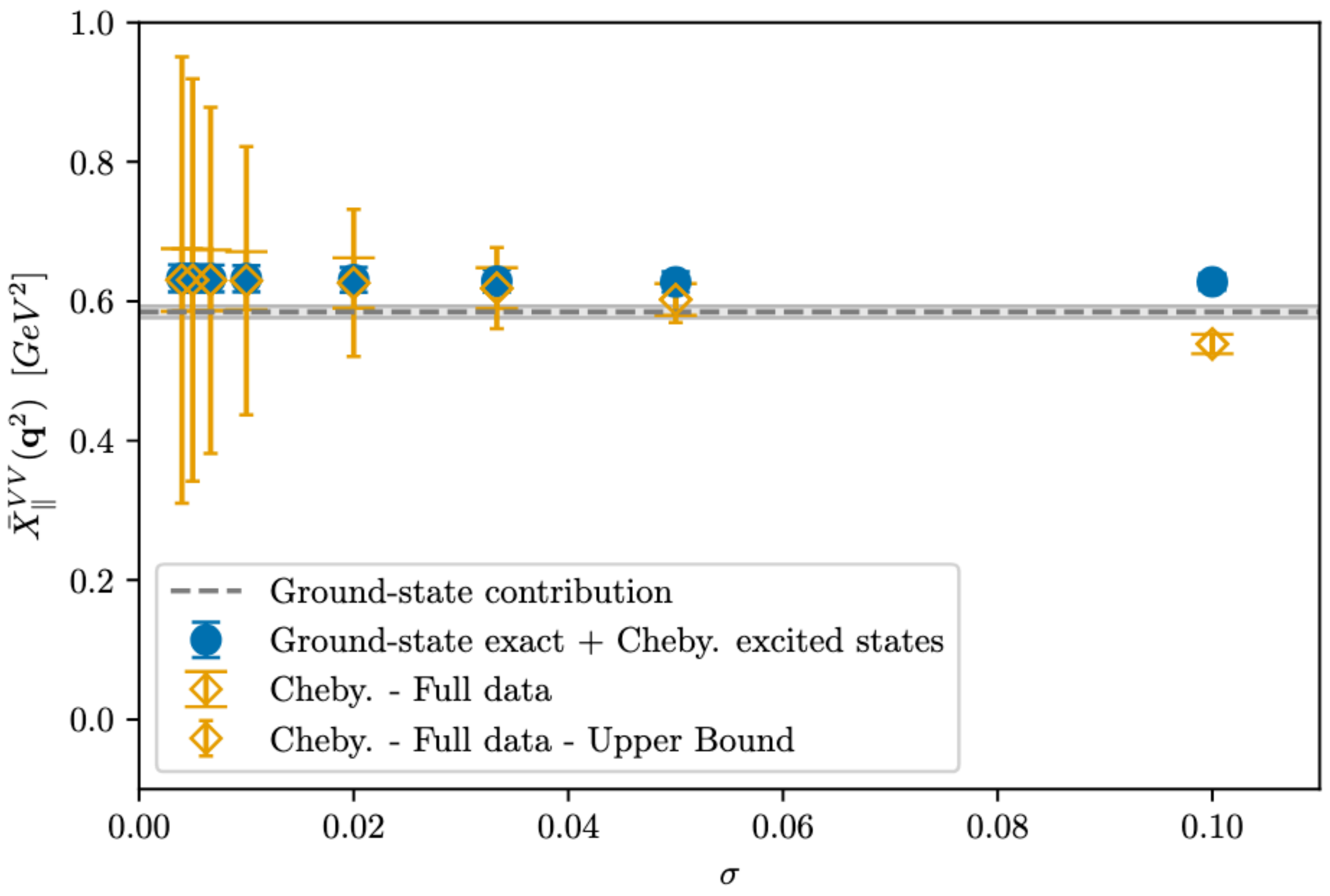
$$\bar{X}^{(l)}(\omega_{\text{th}}) = \int_0^{\boxed{\omega_{\text{th}}}} d\omega\, \rho_{(V)}(\omega)\, K^{(l)}(\omega)$$





Finite-volume effects depend on shape of kernel and quality of approximation

Systematics — $\sigma \to 0$



Example taken for $D_s \to X\ell\bar{\nu}$

[Kellermann et al. arXiv:2504.03358]

- After the infinite-volume limit also the $\sigma \to 0$ limit of vanishing smearing has to be taken
- Here, we assume finite-volume effects are under control (i.e. our data 'is' has been $L \to \infty$ extrapolated)
- estimate potential contribution from higher-order terms
- if we first subtract the ground-state contribution and then apply the inclusive analysis only to the remainder substantially reduces sensitivity to finite smearing width

Summary

Part I: QFT constraints for exclusive meson decays

Unitarity constraint allows to parameterise a limited set of experimental/theory data for form factors in a model- and truncation independent way

Part II: QFT constraints for inclusive meson decays (on the lattice)

Monotonicity and properties of orthogonal polynomials allow to tame the exponential signal-to-noise issue in Euclidean correlators, enabling meaningful predictions for inclusive decay rates

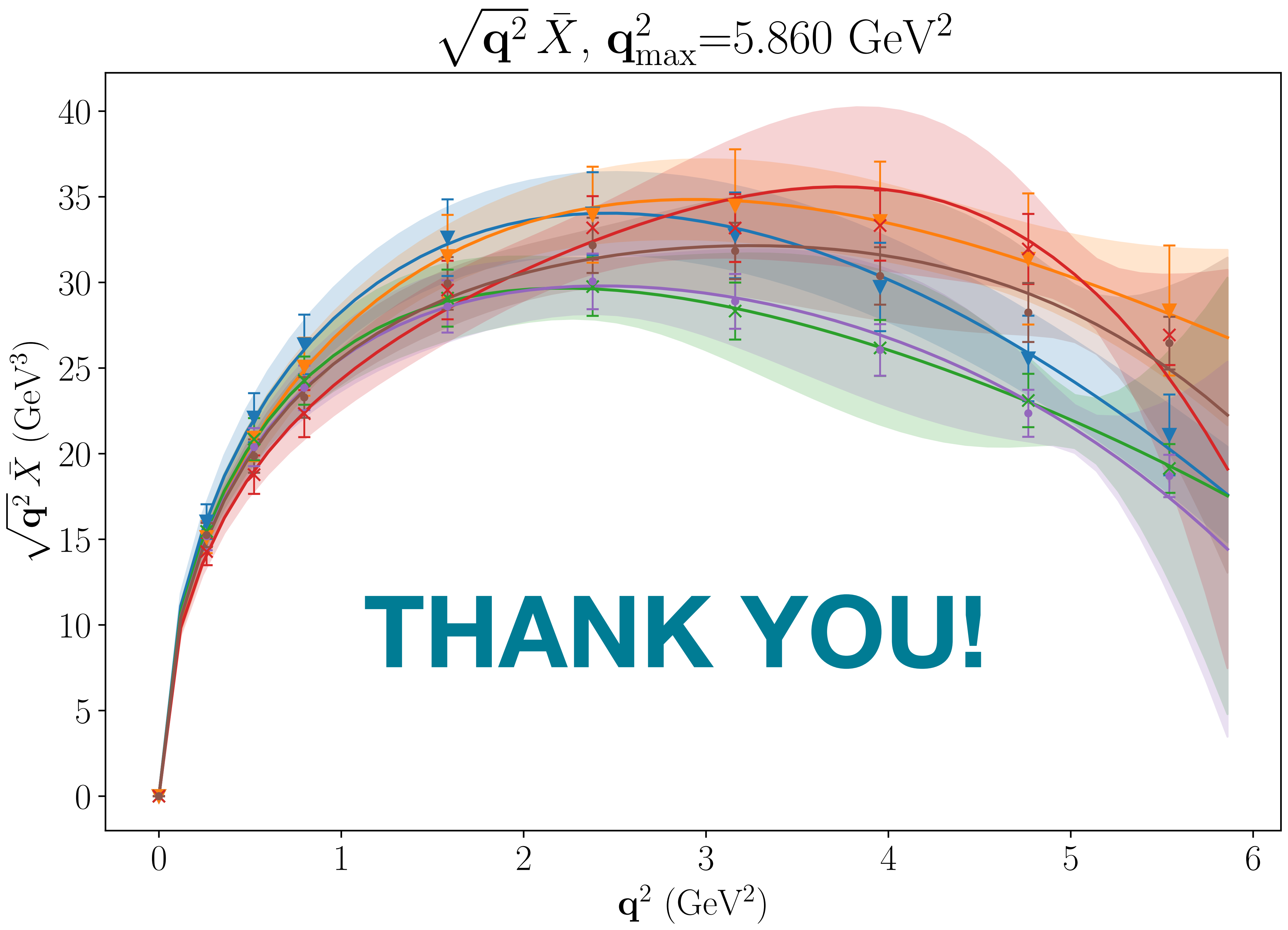
Conclusions

Part I: QFT constraints for exclusive meson decays

- New data will be coming in (LHCb, Belle II, BESIII)
- apply method also to other channels (e.g. $B_s \to K\ell\nu, B \to \pi\ell\nu$, …)
- extend to angular-coefficient analysis like [Martinelli et al. PRD (2025)]

Part II: QFT constraints for inclusive meson decays (on the lattice)

- An independent calculation of $|V_{ub}|$ or $|V_{cb}|$ on the way
- recently new results on inclusive D_s decay [Barone et al. arXiv:2504.03358], [De Santis et al. arXiv:2504.06063, arXiv:2504.06064]
  we can think of novel observables to test continuum techniques / or improved observables to compare with experiment
- We can hopefully soon meaningfully add to the discussion around the $|V_{cb}|$ puzzle


$\sqrt{\mathbf{q}^2}\,\bar{X}$, $\mathbf{q}^2_{\mathrm{max}}$=5.860 GeV^2
$\sqrt{\mathbf{q}^2}\,\bar{X}$ (GeV^3)
$\mathbf{q}^2$ (GeV^2)
THANK YOU!

Bayesian form-factor fit

Flynn, AJ, Tsang, JHEP 12 (2023) 175

Compute BGL parameters as expectation values $\langle g(\mathbf{a})\rangle = \mathcal{N}\int d\mathbf{a}\, g(\mathbf{a})\, \pi(\mathbf{a}\,|\,\mathbf{f}, C_{\mathbf{f}})\, \pi_{\mathbf{a}}$

where *probability for parameters given model and data (assume input Gaussian)*

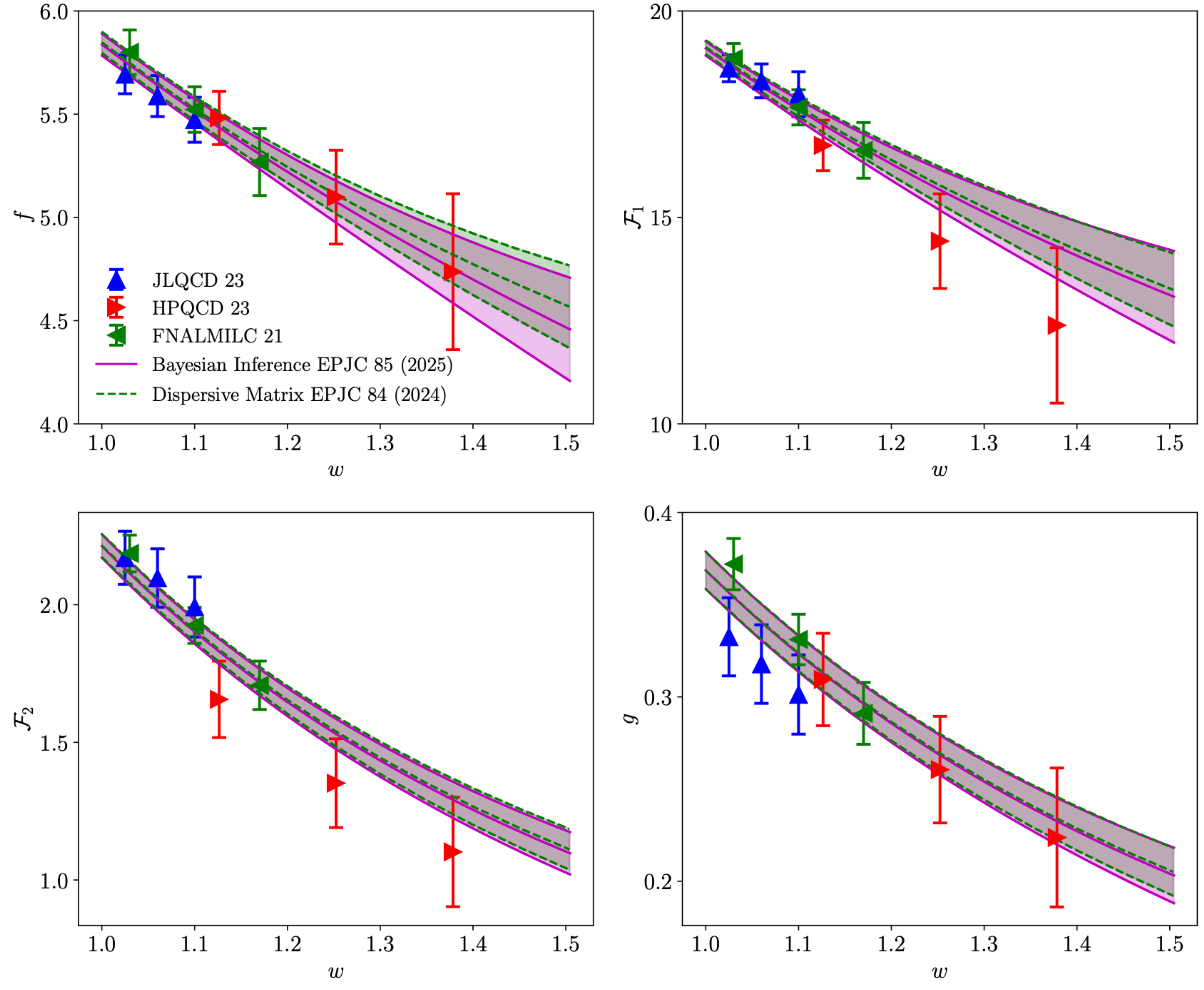
$$\pi(\mathbf{a}\,|\,\mathbf{f}, C_{\mathbf{f}}) \propto \exp\left(-\frac{1}{2}\chi^2(\mathbf{a},\mathbf{f})\right) \quad \text{where} \quad \chi^2(\mathbf{a},\mathbf{f}) = (\mathbf{f}-\mathbf{f}_{\mathrm{BGL}})^T C_{\mathbf{f}}^{-1}(\mathbf{f}-\mathbf{f}_{\mathrm{BGL}})$$

where *prior knowledge is **only** QFT unitarity constraint (flat prior for BGL params):*

$$\pi_{\mathbf{a}} \propto \theta\left(1-|\mathbf{a}_X|^2\right)$$

In practice MC integration: draw samples for **a** from multivariate normal distribution and drop samples not compatible with unitarity

Strategy A: Bayesian Inference vs. Dispersive Matrix



Dispersive-matrix data curtesy of Ludovico Vittorio
(joint talk at Beyond Anomalies 2025)

Bordone, AJ, EPJC (2025)
Martinelli, Simula, Vittorio, EPJC (2024)

- Alternative method: Dispersive Matrix
  [Di Carlo et al. PRD (2021)]
- Excellent agreement between methods
- unitarity and kinematic constraints both relevant and acting in the same way in both approaches, reducing stat. error
- **model-independent parameterisation of form factors successful**
  (insignificant difference probably due to data selection (HPQCD 23))