Resurgence and non-perturbative physics

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The trouble with perturbation theory

Perturbation theory is one of the few universal analytic tools we have in quantum physics. Some of the best tests we have of fundamental theories are based on perturbative series. A famous example is the anomalous magnetic moment of the electron [Kinoshita]:

$$a_e = \frac{1}{2}\left(\frac{\alpha}{\pi}\right) - 0.33\ldots\left(\frac{\alpha}{\pi}\right)^2 + 1.18\ldots\left(\frac{\alpha}{\pi}\right)^3 - 1.91\ldots\left(\frac{\alpha}{\pi}\right)^4 + 6.74\ldots\left(\frac{\alpha}{\pi}\right)^5 + \cdots$$

1 diagram, 7 diagrams, 72 diagrams, 891diagrams, 12672 diagrams

Dyson remarked in 1957 that perturbative series in QED should have zero radius of convergence. More precisely, generic perturbative series in quantum theory diverge **factorially**

$$\varphi(z) = \sum_{n \geq 0} a_n z^n \qquad a_n \sim n!$$

Therefore, they have to be understood as **asymptotic approximations**: given a value of the coupling z, we can provide an approximation to our observable by "optimally truncating" the series. The smaller the value of z, the better the approximation.

This means (among other things) that, if the value of the coupling constant is large enough, knowing more terms in the series does **not** make the approximation better. This is in contrast with convergent series.

Can we improve this state of affairs? Can we "tame" this new divergence in quantum theory?

More conceptually: what is wrong with perturbation theory? What are we missing in the perturbative series?

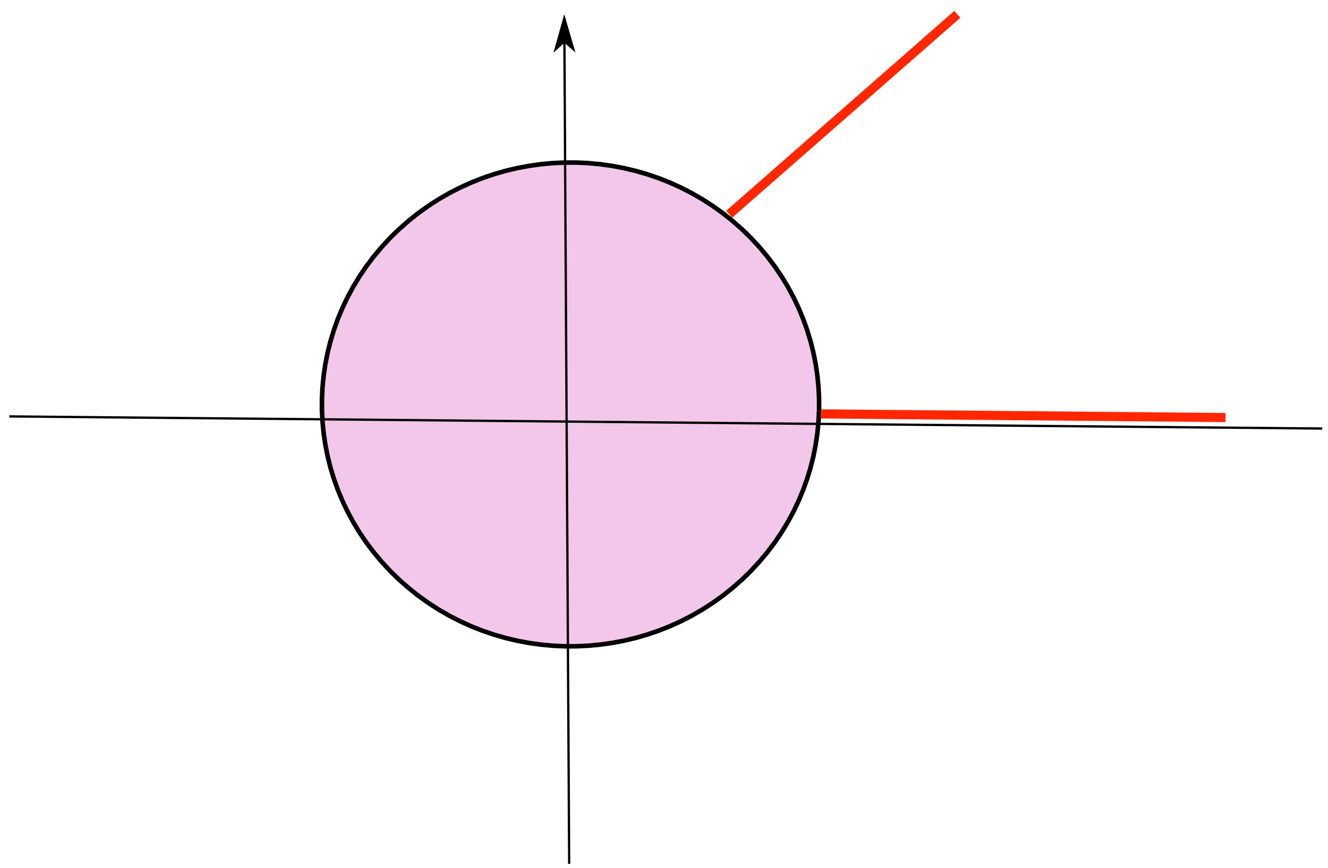
A better mathematical toolbox

It was realized long ago that to improve on this we have to use more sophisticated mathematical tools to manage perturbative series. Of these, the best one seems to be **Borel resummation**. This starts by a radical surgery operation called **Borel transform**:

$$\varphi(z) = \sum_{n\geq 0} a_n z^n \xrightarrow{\text{Borel transform}} \widehat{\varphi}(\zeta) = \sum_{n\geq 0} \frac{a_n}{n!} \zeta^n$$

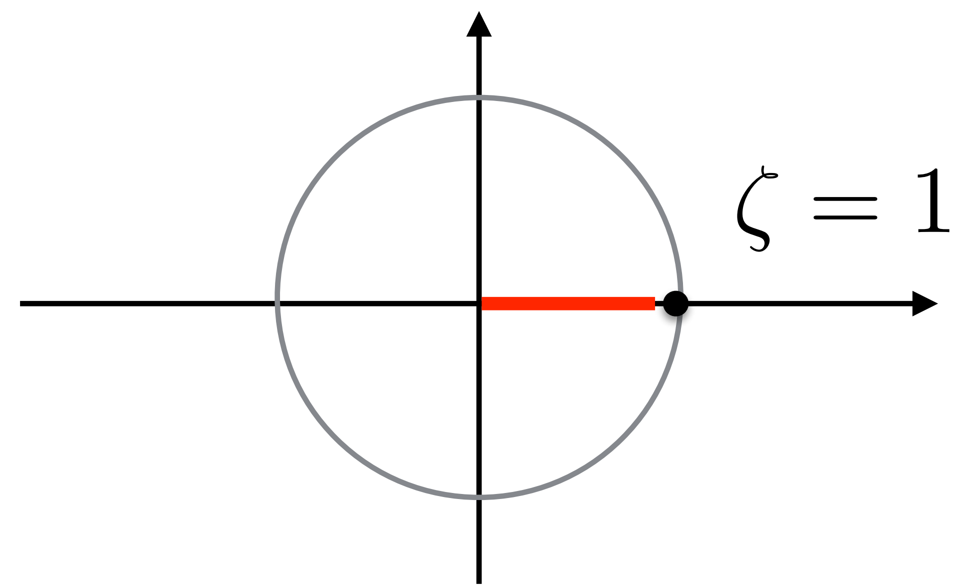
We just cut out the factorial growth!

The Borel transform $\widehat{\varphi}(\zeta)$ is **analytic** at the origin. Very often it can be analytically continued to the complex plane, displaying **singularities** (poles, branch cuts).



Example: $$\varphi(z) = \sum_{k\geq 0} k! z^k$$

$$\widehat{\varphi}(\zeta) = \sum_{k\geq 0} \zeta^k = \frac{1}{1-\zeta}$$



Note that the Borel transform takes us from the world of divergent formal power series to the world of analytic functions, which is perhaps the nicest world in mathematics

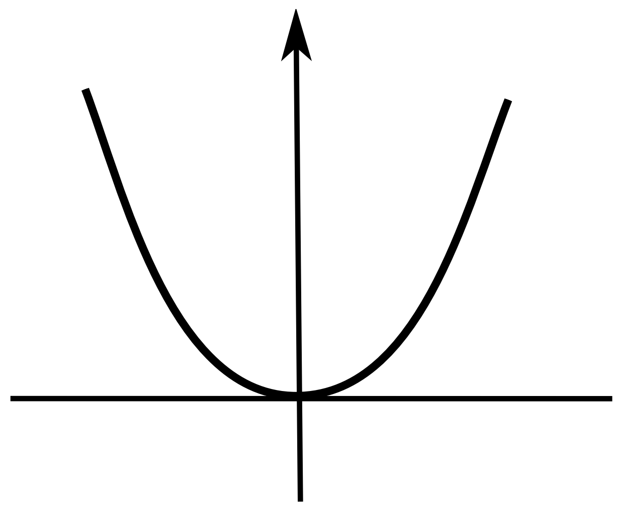
The Borel resummation of the original series is essentially the Laplace transform of its Borel transform:

$$s(\varphi)(z) = z^{-1} \int_0^\infty \mathrm{e}^{-\zeta/z} \widehat{\varphi}(\zeta) \mathrm{d}\zeta$$

If it exists, its asymptotic expansion for small *z* coincides with the original series

divergent series $\longrightarrow$ function

In some cases Borel resummation works remarkably well, as in the energies of the quartic oscillator in quantum mechanics



$$\mathsf{H} = \frac{\mathsf{p}^2}{2} + \frac{\mathsf{x}^2}{2} + g\mathsf{x}^4$$

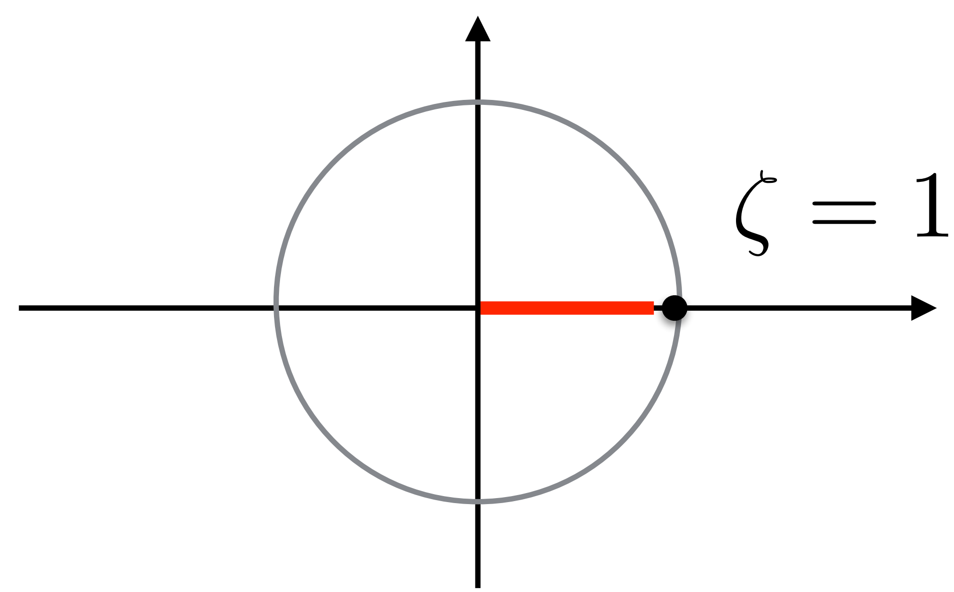
$$E_0(g) \sim \frac{1}{2} + \frac{3}{4}g - \frac{21}{8}g^2 + \cdots$$

Here, Borel resummation of the perturbative series in the coupling constant reproduces the **exact** spectrum, level by level [Graffi-Grecchi-Simon]

However, there are cases where the procedure does not work:

If the Borel transform has singularities along the positive real axis, the integral giving the resummation is ill-defined, and the series is not Borel summable.

$$\varphi(z) = \sum_{k \geq 0} k! z^k$$



$$\int_0^\infty \frac{\mathrm{e}^{-\zeta/z}}{1-\zeta} \mathrm{d}\zeta = ???$$

The “old” theory of Borel resummation stopped here: if your series is Borel resummable, proceed and calculate. If it is not, abandon all hope.

But what is the meaning of these singularities obstructing Borel resummation?

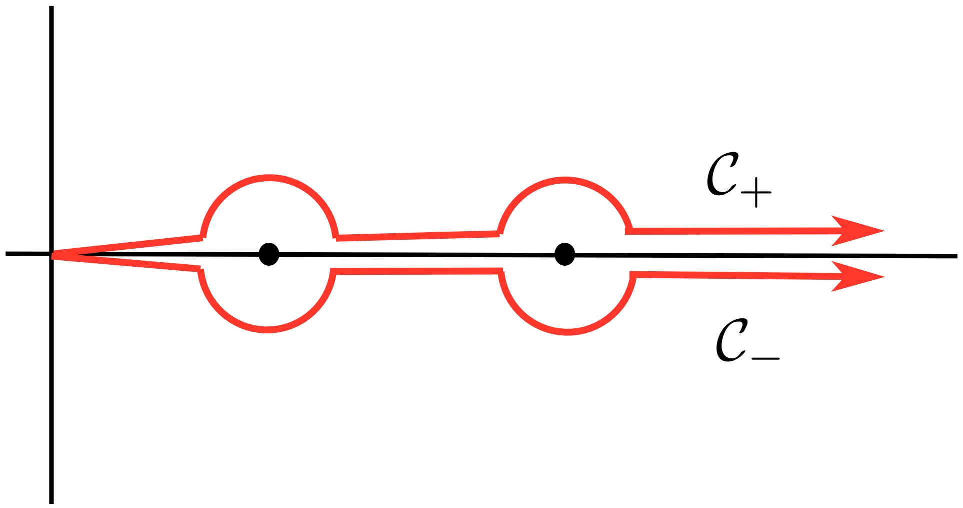
Answer: these singularities are the smoking gun of **non-perturbative corrections**, beyond what is obtained in perturbation theory

In other words, non-perturbative effects beyond perturbation theory, **resurge** in the singularities of the Borel transform!

Detecting non-perturbative effects

How do we detect these effects? One possibility is to use the "radar method" due to André Voros

Instead of hitting the singularities, deform the contours to obtain **lateral Borel resummations**



$$s_{\pm}(\varphi)(z) = z^{-1} \int_{\mathcal{C}_{\pm}} \mathrm{e}^{-\zeta/z} \widehat{\varphi}(\zeta) \mathrm{d}\zeta$$

If there is a singularity at $\zeta = A$, the two lateral resummations differ by an imaginary, **exponentially small quantity**

$$s_+(\varphi)(z) - s_-(\varphi)(z) \sim \mathrm{i}\, z^{-b} \mathrm{e}^{-A/z} \sum_{n\geq 0} c_n z^n$$

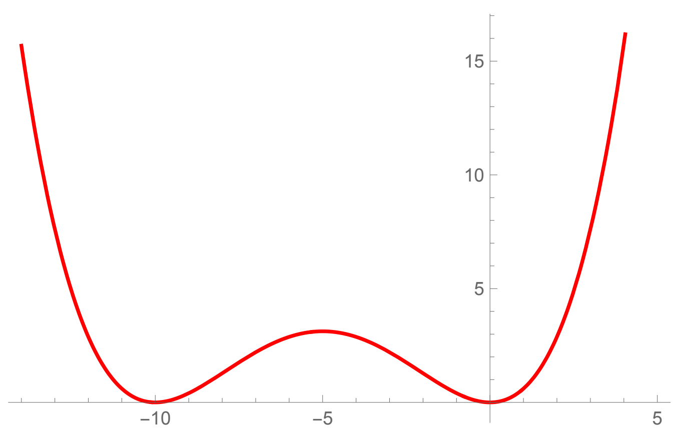
Therefore, we find **hidden power series** at the singularities, with exponentially small prefactors. These series signal the existence of additional, non-perturbative sectors of the theory

Equivalently, one can detect these sectors in the **large order behavior** of perturbation theory [Bender-Wu]

$$a_n \sim A^{-n-b} \Gamma(n+b) \left(c_0 + \frac{c_1 A}{n+b-1} + \cdots \right)$$

An example in quantum mechanics

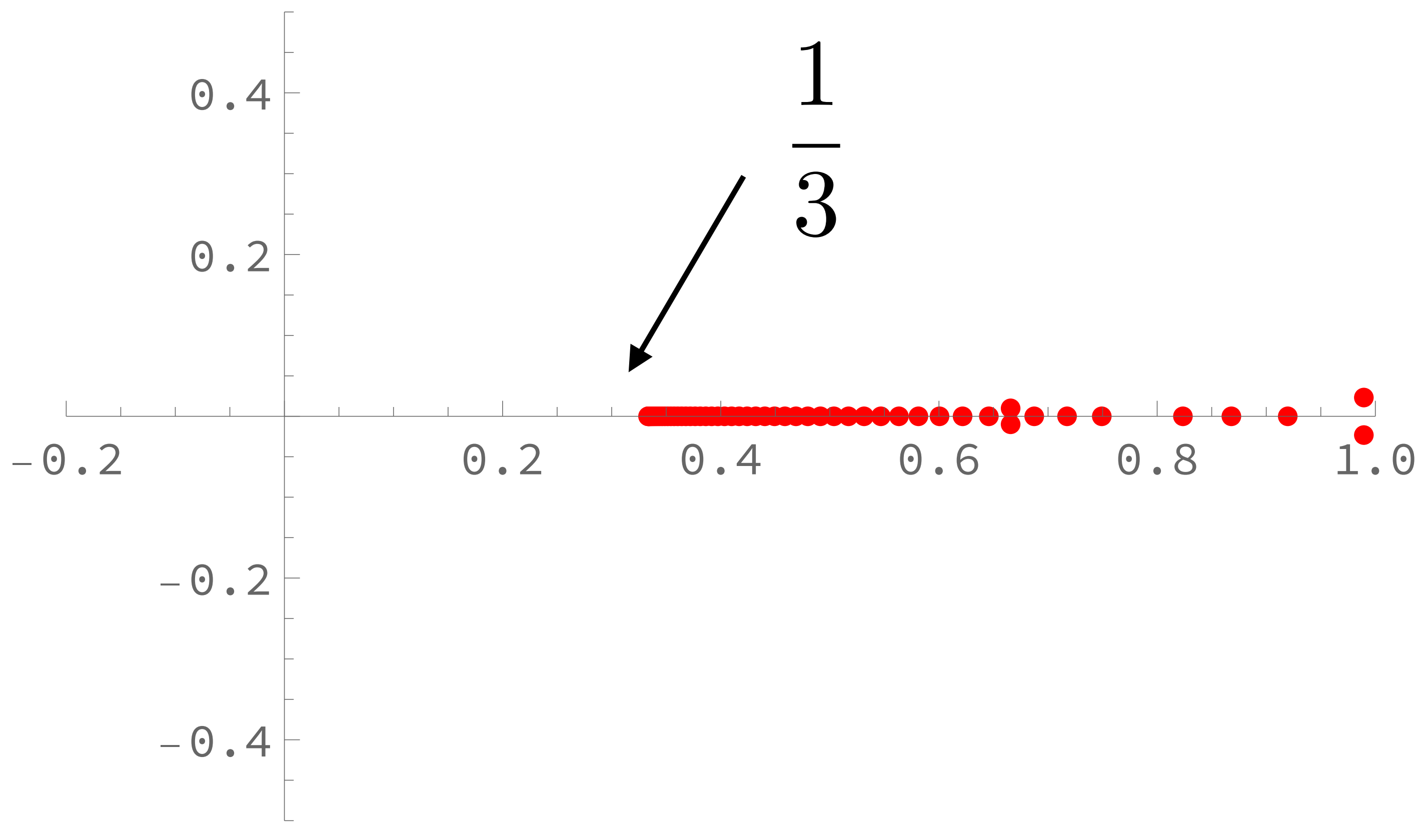
To illustrate this idea, let us consider a simple example in quantum mechanics: the double well potential



$$V(x) = \frac{1}{2}x^2(1+gx)^2$$

$$E_0(g) = \sum_{k\geq 0} a_k g^{2k} = \frac{1}{2} - g^2 - \frac{9g^4}{2} - \frac{89g^6}{2} - \frac{5013g^8}{8} - \cdots$$

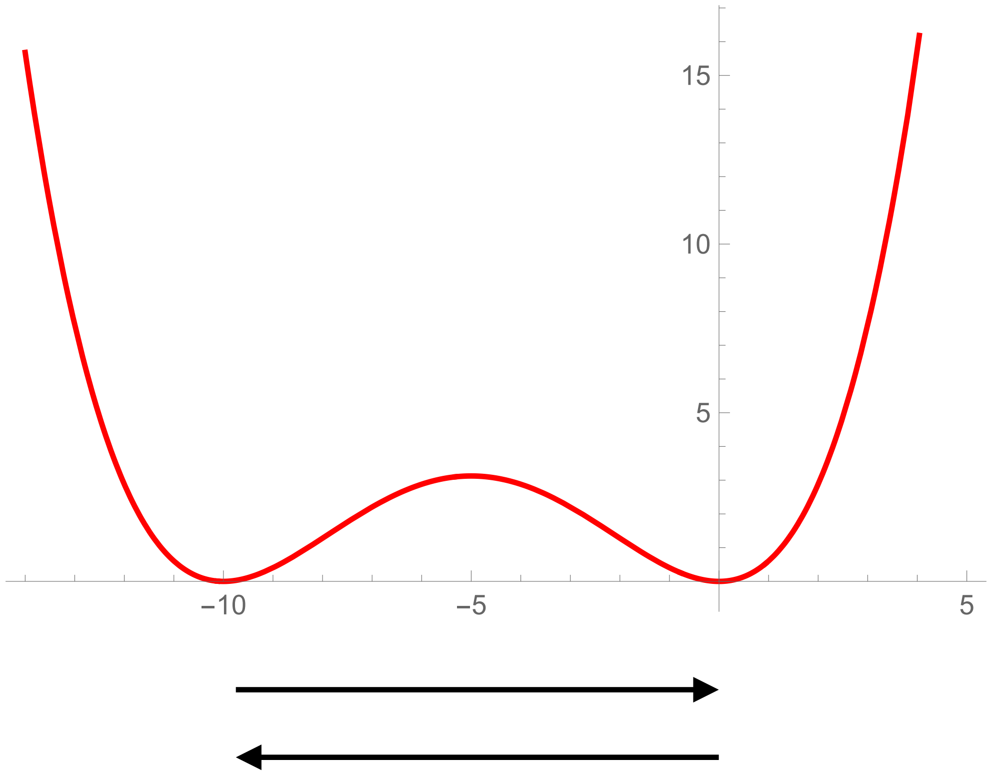
The perturbative series for the ground state energy is **not** Borel summable



If we use the "radar method", the discontinuity of the Borel resummation gives us the series

$$\frac{1}{g^2} e^{-\frac{1}{3g^2}} \left(1 - \frac{53g^2}{6} - \frac{1277g^4}{72} - \dots \right)$$

This series unveils another sector of the theory! It corresponds to the expansion around a different background, given by an **instanton** (i.e. a classical solution of the equations of motion) interpolating between the two vacua



From perturbative to exact

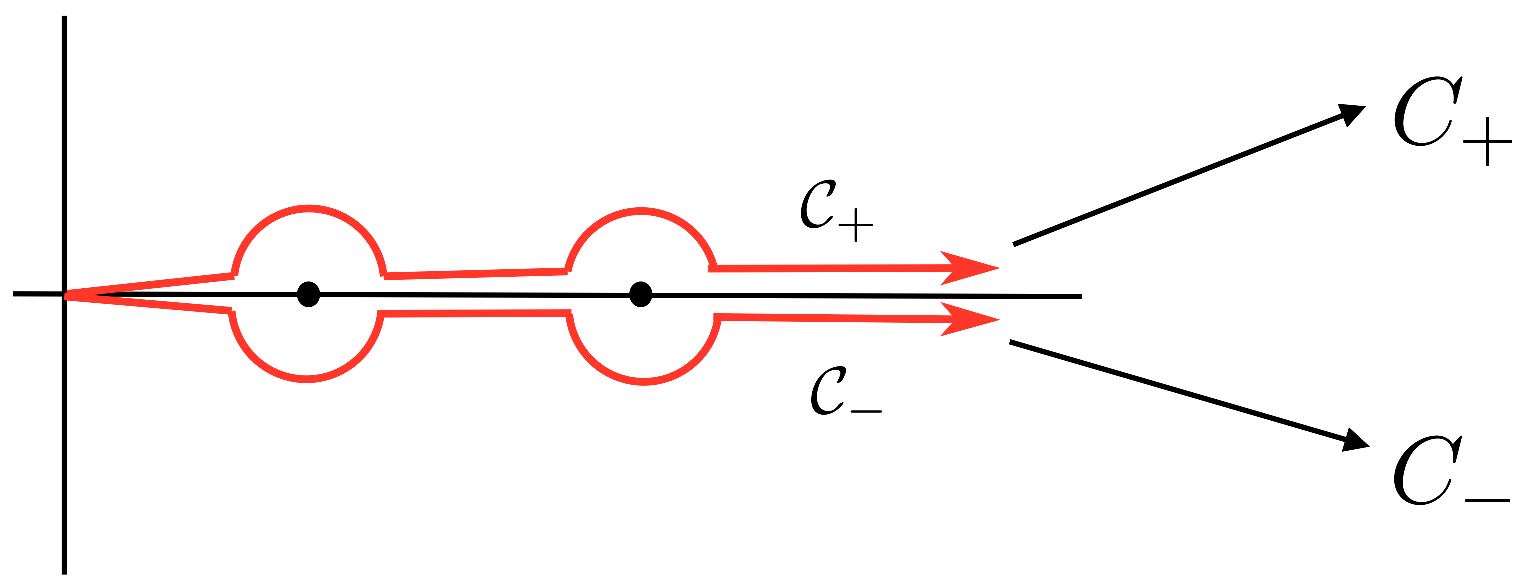
This is all very interesting, but what can we do with it?

These additional sectors give us **exponentially small corrections** to physical observables. We should then promote our perturbative series to a **trans-series**:

$$\varphi(z) + C\mathrm{e}^{-A/z} z^{-b} \sum_{n\geq 0} c_n z^n + \cdots$$

This is a formal object, but we can obtain actual numbers by doing (lateral) Borel resummations

Non-Borel summability is not a problem anymore. We just have to correlate the value of the prefactor *C* with the choice of lateral resummation, so that the final result is physical. To obtain a real answer, for example, *C* might have to be complex!



Therefore, in a theory with non-Borel summable series, the strength of a non-perturbative correction is **ambiguous** and is generically **complex** [Dingle, Voros, Silverstone, David, …]

The research program of resurgence is based on the idea that

all observables in a quantum theory which have an asymptotic expansion can be obtained exactly by (lateral) Borel resummations of trans-series

Pros and cons

Potentially, this gives a universal definition of non-perturbative physics built upon perturbative tools

“Semiclassical decoding”: in contrast to e.g. numerical methods, the building blocks of trans-series can be understood analytically as contributions from the perturbative sector, plus a series of non-perturbative corrections

“It is nice to know that the computer understood the answer, but I would like to understand it too”

Beautiful mathematics

Very good at detecting the presence of non-perturbative effects

Too universal a language! It does not give concrete tools to e.g. obtain trans-series. These have to be found elsewhere

Works better when we have a lot of control of perturbation theory, which is not often the case

Not yet clear if resurgence is a discovery tool or an organizational tool

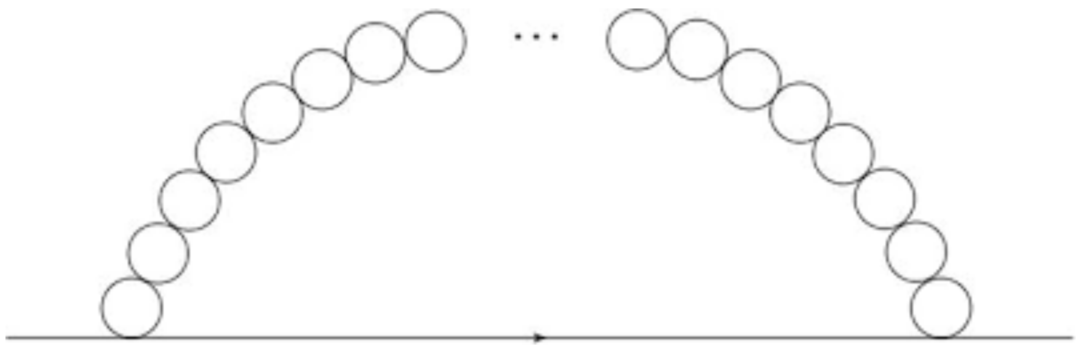
Where do non-perturbative effects come from?

As in our quantum-mechanical example, a typical source of trans-series in quantum theories are expansions around different saddle points (i.e. **instanton sectors**) of the path integral.

For a while, people had the hope that trans-series in QFTs would come just from perturbative and instanton sectors. Sometimes this is the case, like the $\lambda\Phi^4$ theory in two dimensions [Serone et al.]

However, the dream of instanton dominance was shattered in the 1970s-1980s by the discovery of a new and mysterious source of non-perturbative effects: **renormalons**

One manifestation of renormalons are bubble diagrams in renormalizable theories, which grow factorially at large order due to the integration over momenta.



$$\int_0^1 (-\log(k))^n \, \mathrm{d}k = n!$$

Renormalon effects play a very important role in e.g. gauge theories. They are believed to provide the most important non-perturbative effects in QED and QCD, and to explain the factorial divergence of their series

In QCD (and more generally, in asymptotically free theories), they are believed to lead to **power corrections** of the form

$$\mathrm{e}^{-\frac{d}{2|\beta_0|g(\mu)}} \sim \left(\frac{\Lambda}{\mu}\right)^d$$

β_0
first coeff.
beta function

Renormalons do not have a known description in terms of saddle-points of the path integral, and their calculation from first principles remains a challenge.

Although the ideas of resurgence have been (implicitly) used to understand renormalons in QCD, verifying them in detail proves difficult. One can however study renormalons in “toy models” of asymptotically free theories, like the 2d non-linear sigma model and the 2d Gross-Neveu model.

By exploiting that these models are **integrable** one can determine renormalon corrections analytically, and without relying on the large *N* approximation typically used in renormalon physics. One finds in addition counterexamples to renormalon folklore [M.M.-Miravitllas-Reis, Bajnok-Balog et al.].

The observables under control are **exactly** reproduced by the resummation of trans-series incorporating renormalon effects

Conclusions

Resurgence is, first of all, the art of working with divergent series, and it suggests that conventional perturbation theory should be replaced by "resurgent perturbation theory", in which perturbative series are replaced by trans-series. We expect that many physical observables can be "decoded" in terms of trans-series.

There are two known sources for trans-series in quantum theory: instantons, which are relatively well-understood, and renormalons, which we don't know how to incorporate in the path integral. The calculation from first principles of trans-series associated to renormalons remains one of the most crucial open problems.

Thank you for your attention