Lattice QCD+QED calculations for high-precision tests of the Standard Model

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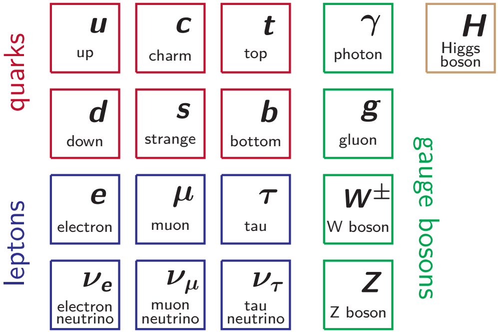
The Standard Model of Particle Physics



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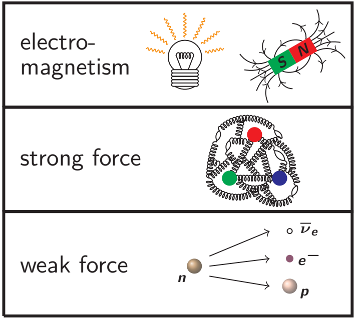
leptons	e electron	μ muon	τ tau	$W^{\pm}$ W boson	
	ν_e electron neutrino	ν_μ muon neutrino	ν_τ tau neutrino	Z Z boson	



electro-magnetism

strong force

weak force

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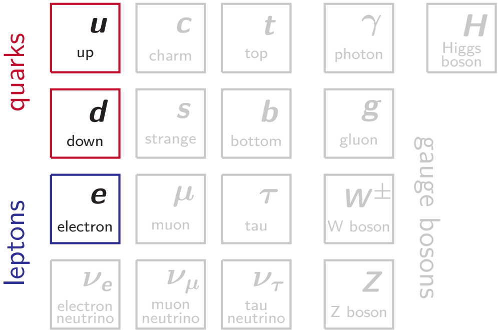

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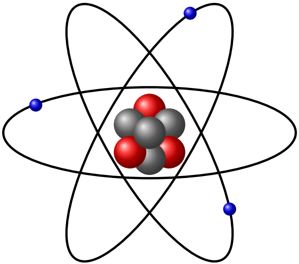

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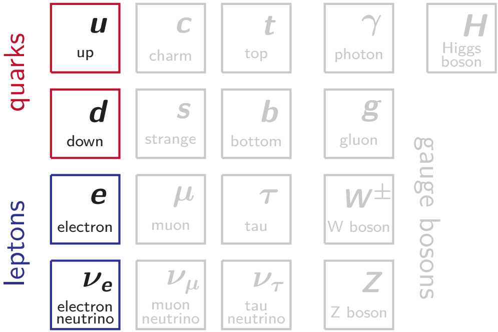

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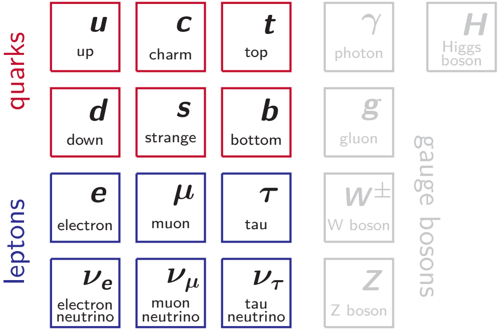

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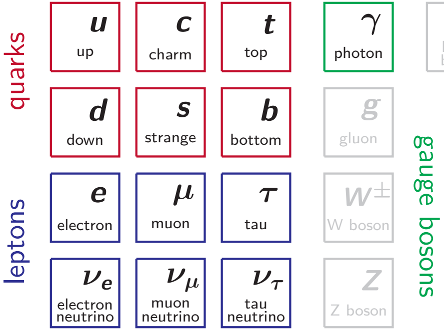

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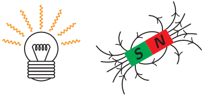

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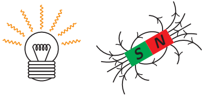
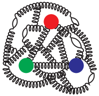


strong force



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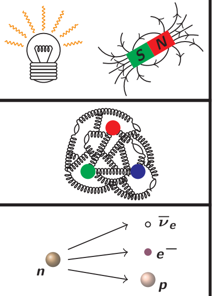

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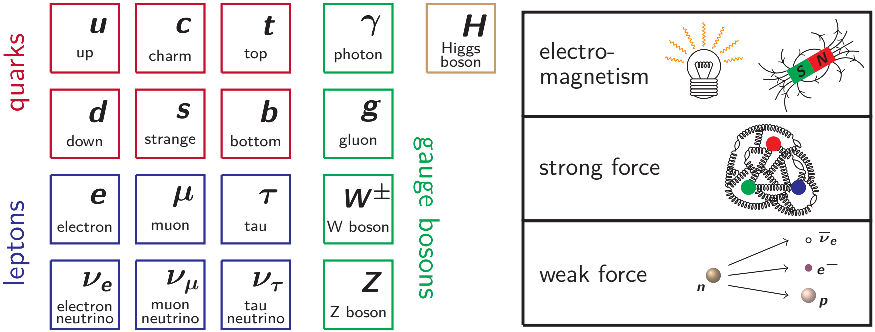


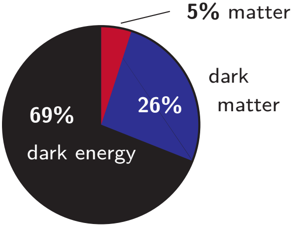
Higgs Boson discovered at LHC in 2012

→ Noble Prize 2013 for P. Higgs and F. Englert for prediction

The need for new physics

Many unsolved questions, for example:

- What is dark matter? Or dark energy?



- Why is there more matter than antimatter in the Universe?
- Why are there three generations of fermions?
- . . .

New physics is out there!

Hunting for new physics

High-Energy Frontier:

Searches at particle colliders such as the LHC at CERN

[https://cds.cern.ch/record/1295244]

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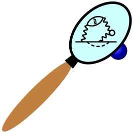
High-Precision Frontier:

New physics contributes via quantum effects



"The closer you look
the more there is to see"
[F. Jegerlehner, The Anomalous Magnetic Moment of the Muon]

- precision measurements of properties of known particles
- precise calculation within the Standard Model

→ find (potential) discrepancies

QCD at low energies

- Quantum Chromo Dynamics (QCD)
  → theory of the strong interaction
- strong coupling $\alpha_s \sim \mathcal{O}(1)$ at small energies
- quarks and gluons confined to hadrons

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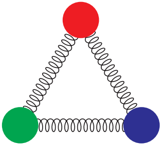

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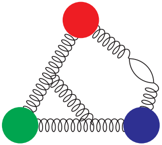

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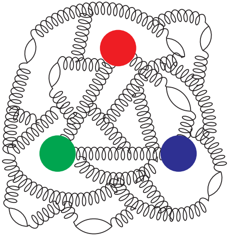

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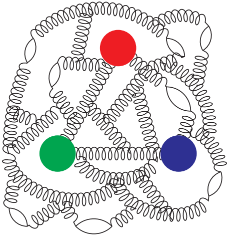

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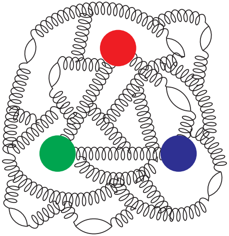

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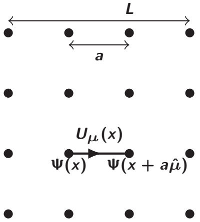
Lattice QCD in a nutshell



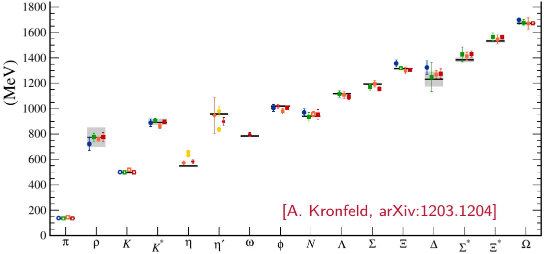
- Discretize (Euclidean) space-time by a 4d lattice
- Quantize QCD using Euclidean path integrals

$$\langle A\rangle = \frac{1}{Z}\int \mathcal{D}[\Psi,\overline{\Psi}]\mathcal{D}[U]\, e^{-S_E[\Psi,\overline{\Psi},U]}\, A(U,\Psi,\bar{\Psi})$$

- gluonic expectation value: Monto Carlo techniques
- extrapolate to $a \to 0$ and $L \to \infty$

Overview Lattice QCD calculations

- Spectrum of hadron masses

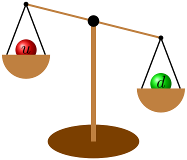


[A. Kronfeld, arXiv:1203.1204]

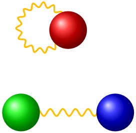
- Quantities studied in Lattice calculations include
  - Hadron Spectroscopy & interactions
  - weak decays of Hadrons
  - Hadron Structure
  - hadronic contributions to Muon g-2
  - QCD phase diagram
  - ...
- many lattice calculations reaching prescision $\lesssim$ **1%**

Isospin Breaking Corrections

- lattice calculations usually in the isospin symmetric limit "$\boldsymbol{u = d}$"
- two sources of isospin breaking effects
  - different masses for up- and down quark $\mathcal{O}((m_d - m_u)/\Lambda_{\mathrm{QCD}})$



  - Quarks have electric charge $\mathcal{O}(\alpha)$



- lattice calculation with $\lesssim$ **1%** precision requires isospin breaking

Adding QED to the lattice calculation

- Euclidean path integral including QED

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}[\Psi, \overline{\Psi}] \mathcal{D}[U] \mathcal{D}[A] \; O \; e^{-S_F[\Psi, \overline{\Psi}, U, A]} \; e^{-S_G[U]} \; e^{-S_\gamma[A]}$$

- perturbative expansion in α [RM123 Collaboration, Phys.Rev. **D87**, 114505 (2013)]

$$\langle O \rangle = \langle O \rangle_{e=0} + \frac{1}{2} e^2 \frac{\partial^2}{\partial e^2} \langle O \rangle \bigg|_{e=0} + \mathcal{O}(\alpha^2)$$

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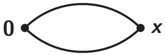
- example: pion two-point function $\left\langle \phi(x) \phi^\dagger(0) \right\rangle$ with $\phi^\dagger = \overline{u} \gamma_5 d$

$$\left\langle \phi(x) \phi^\dagger(0) \right\rangle_{e=0} =$$



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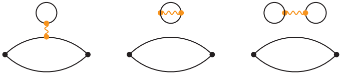

QED corrections to sea-quarks

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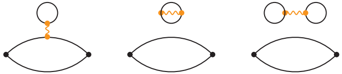


QED corrections to sea-quarks

- Finite volume corrections, usually $\sim \frac{1}{(m_\pi L)^n}$

Mass shifts and strong IB corrections

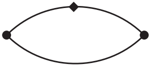
- (bare) quark masses are free parameters of QCD
- option 1: Calculate directly with different up- and down-quark masses
- option 2: perturbative expansion in $\boldsymbol{\Delta m_q = (m_q^0 - m_q)}$

  [G.M. de Divitiis *et al*, JHEP 1204 (2012) 124]

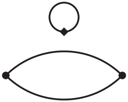
$$\langle O\rangle_{m_q} = \langle O\rangle_{m_q^0} + \Delta m_q \frac{\partial}{\partial m_q}\langle O\rangle\Big|_{m_q^0} + \mathcal{O}\left(\Delta m_q^2\right)$$

  $\longrightarrow$ insertions of scalar currents $\overline{q}\mathbb{1}q$

- e.g., for meson two-point function



sea quark effects:
quark-disconnected
diagrams



Tuning to physical point in a QCD+QED, $m_u \neq m_d$

- choose input quark masses such that a set of hadron masses receive their experimentally measured value including QED, e.g.

$$(m_u,\, m_d,\, m_s,\, \dots) \qquad \longrightarrow \qquad (M_{\pi^+},\, M_{K^+},\, M_{K^0},\, \dots)$$

$$\begin{aligned}
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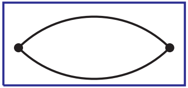

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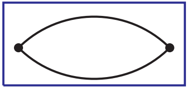

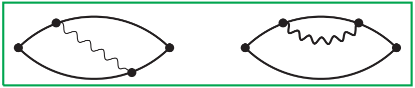

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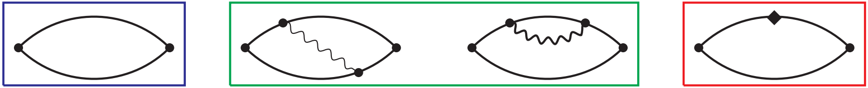

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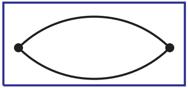
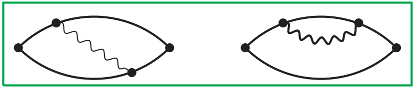
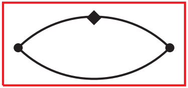






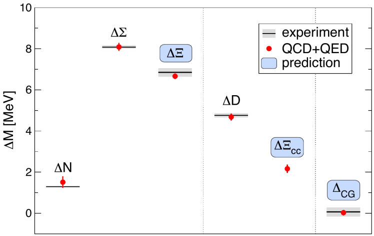
(plus another mass to determine the scale, e.g. the Ω-Baryon)

Overview lattice calculations of IB corrections

Hadron Masses

- Various calculations on IB corrections to hadron masses
- e.g., proton-neutron mass splitting

[Sz. Borsanyi *et al*, Science 347 (2015) 1452-1455]



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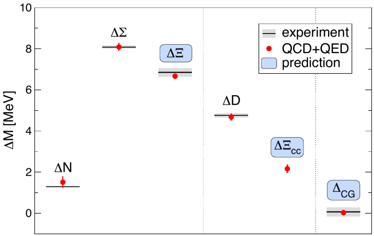
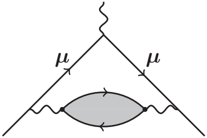


Muon g-2

- Hadronic Vacuum Polaristion contribution to g-2



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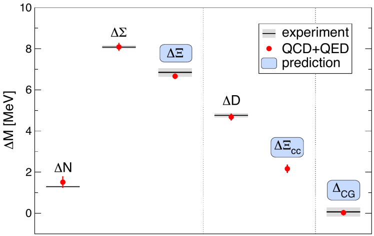

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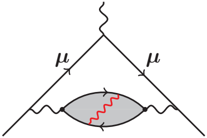

- first calculations [VG *et al*, JHEP 09 (2017) 153; VG *et al*, Phys. Rev. Lett. 121 022003 (2018)]
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Overview lattice calculations of IB corrections

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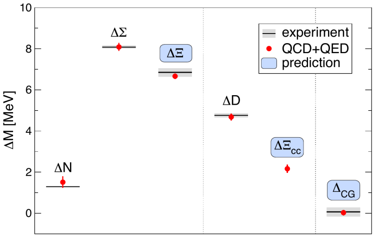

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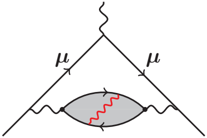

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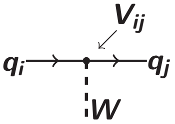
- first calculations for leptonic meson decays

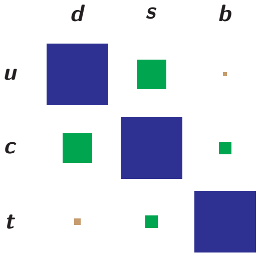
Quark-mixing CKM matrix

u up	c charm	t top
d down	s strange	b bottom

- charged weak interaction ($W^\pm$): changes "up"-type quarks into a "down"-type quarks
- mixes different generation of quarks
- quark-mixing Cabibbo–Kobayashi–Maskawa (CKM) matrix

$$V_{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

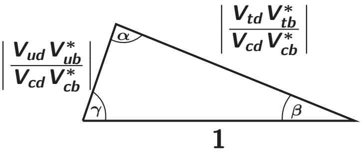


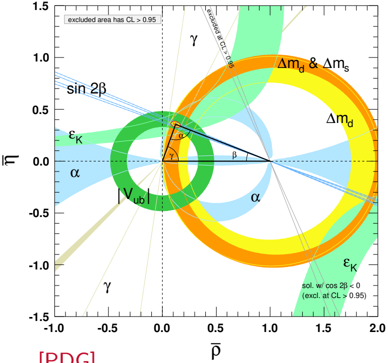


Unitarity of the CKM matrix

- within the Standard Model CKM matrix is unitary $\boldsymbol{V}_{\text{CKM}}\boldsymbol{V}_{\text{CKM}}^{\dagger} = \mathbb{1}$
  $\rightarrow$ test the Standard Model by finding deviations from Unitarity
- example

$$\boldsymbol{V_{ud}V_{ub}^{*} + V_{cd}V_{cb}^{*} + V_{td}V_{tb}^{*} = 0}$$

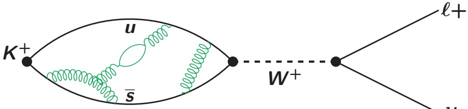




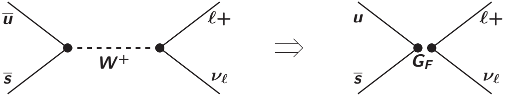
[PDG]

V_{us} from leptonic Kaon decays

- leptonic Kaon decay $K^+ \to \ell^+\nu_\ell$



- effective weak Hamiltonian



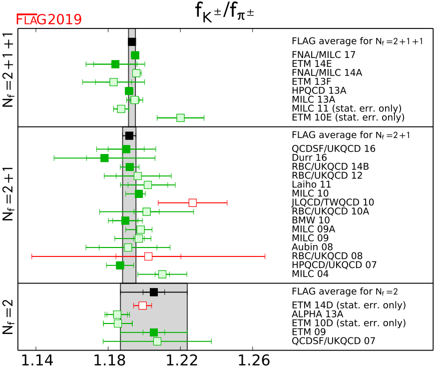
- decay rate (can be measured experimentally)

$$\Gamma(K^+ \to \ell^+\nu_\ell) = \frac{G_F^2\,|V_{us}|^2\,f_K^2}{8\pi}\,M_K\,m_\ell^2\left(1-\frac{m_\ell^2}{M_K^2}\right)^2$$

- known factors (Fermi constant G_F, masses m)
- CKM matrix element V_{us}
- kaon decay constant f_K, can be calculated on the lattice

f_K/f_π from the lattice

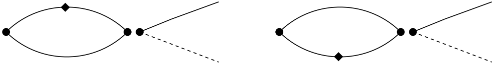
- leptonic meson decays $K/\pi \to \ell\nu_\ell$
- pseudoscalar meson decay constant from the lattice
- overview Kaon/Pion decay constants



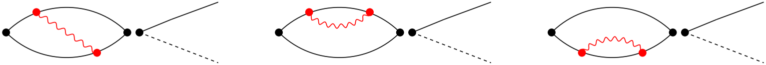
- results with precision < **1%**

perturbative expansion - leptonic meson decay

- total decay rate $\boldsymbol{\Gamma(P^+ \to \ell^+\nu_\ell) = \Gamma^0 + \delta\Gamma = \Gamma^0(1+\delta R)}$
- mass-shift corrections $\boldsymbol{\mathcal{O}(\Delta m)}$



- quark QED corrections $\boldsymbol{\mathcal{O}(e_q^2)}$



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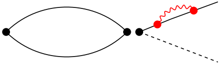

- quark-lepton QED correction $\boldsymbol{\mathcal{O}(e_\ell e_q)}$

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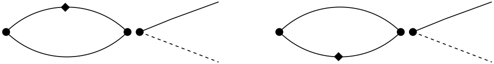

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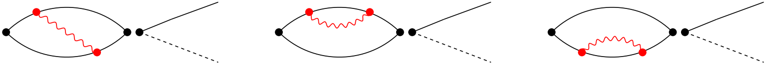

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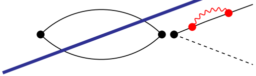

$\longrightarrow$ absorbed in renormalisation of lepton

- quark-lepton QED correction $\mathcal{O}(e_\ell e_q)$

perturbative expansion - leptonic meson decay

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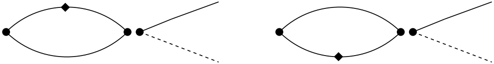

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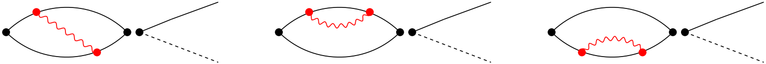

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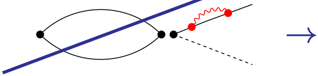

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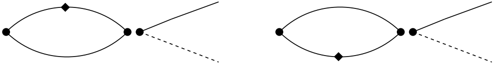

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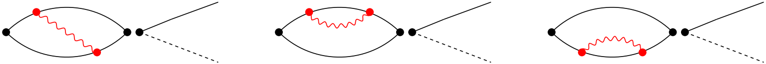

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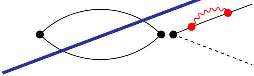


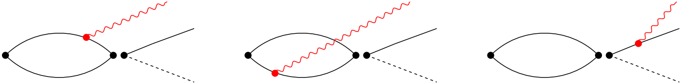
→ absorbed in renormalisation of lepton

- quark-lepton QED correction $\boldsymbol{\mathcal{O}(e_\ell e_q)}$

IB corrections to leptonic meson decay

- Infrared divergencies cancled by diagrams with one final state photon

$$\Gamma(K^+ \to \ell^+\nu_\ell, \alpha) + \Gamma(K^+ \to \ell^+\nu_\ell\gamma)$$



- formalism developed in [N. Carrasco *et al*, Phys.Rev. **D91**, 074506 (2015)]
- real photon emission:
  - for small (enough) energy of final state photon
    → cannot resolve structure of meson
    → calculate analytically in point-like approximation for meson
  - can also be calculated on the lattice [A. Desiderio *et al*, Phys.Rev.D 103 (2021) 1, 014502], [D. Giusti *et al*, Phys.Rev.D 107 (2023) 7, 074507]
- first lattice results for IB corrections to π, K decays
  - RM123-Soton [M. Di Carlo *et al*, Phys.Rev.D 100 (2019) 3, 034514], [D. Giusti *et al*, Phys. Rev. Lett.**120**, 072001 (2018)]
  - RBC/UKQCD (first physical mass calculation) [VG *et al*, JHEP 02 (2023) 242]

RBC/UKQCD collaboration

UC Berkeley/LBNL

Aaron Meyer

University of Bern & Lund

Nils Hermansson Truedsson

BNL and BNL/RBRC

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Peter Boyle (UoE)
Taku Izubuchi
Chulwoo Jung
Christopher Kelly
Meifeng Lin
Nobuyuki Matsumoto
Shigemi Ohta (KEK)
Amarjit Soni
Tianle Wang

CERN

Andreas Jüttner (Southampton)
Tobias Tsang

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Norman Christ
Yikai Huo
Yong-Chull Jang
Joe Karpie
Bob Mawhinney
Bigeng Wang
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Douglas Stewart
Joshua Swaim
Masaaki Tomii

Edinburgh University

Matteo Di Carlo
Luigi Del Debbio
Felix Erben
Vera Gülpers
Maxwell T. Hansen
Tim Harris
Ryan Hill
Raoul Hodgson
Nelson Lachini
Michael Marshall
Fionn Ó hÓgáin
Antonin Portelli
James Richings
Azusa Yamaguchi
Andrew Yong

University of Liverpool

Nicolas Garron

Michigan State University

Dan Hoying

Milano Bicocca

Mattia Bruno

Nara Women's University

Hiroshi Ohki

Peking University

Xu Feng

University of Regensburg

Davide Giusti
Christoph Lehner (BNL)

University of Siegen

Matthew Black
Oliver Witzel

University of Southampton

Alessandro Barone
Jonathan Flynn
Nikolai Husung
Rajnandini Mukherjee
Callum Radley-Scott
Chris Sachrajda

Stony Brook University

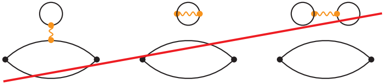
Jun-Sik Yoo
Sergey Syritsyn (RBRC)

VG *et al*, JHEP 02 (2023) 242, arXiv:2211.12865

Details of the calculation

- $N_f = 2 + 1$ flavours of Möbius Domain Wall Fermions
- $L^3 \times T = 48^3 \times 96$ with inverse lattice spacing of $a^{-1} = 1.73$ GeV
- simulation at (close-to) physical quark masses

- Feynman gauge and QED_L for the photon
- electro-quenched approximation (sea-quarks are neutral)

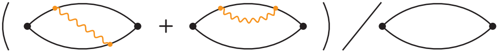


- non-factorisable diagram to leptonic decay

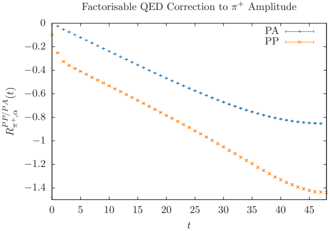
  - include lepton (with mass of muon) on the lattice
  - twisted boundary conditions to tune momentum of muon such that energy and momentum are conserved
- real photon emission analytically for point-like mesons

Results - Lattice Correlators (Plots by A. Yong)

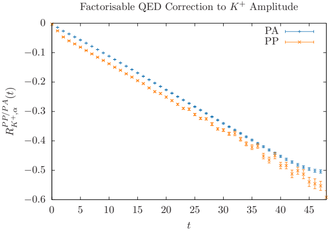
- factorisable contributions



$$\sim (\boldsymbol{A} - \boldsymbol{\Delta M_P\, t})$$



Factorisable QED Correction to π^+ Amplitude



Factorisable QED Correction to K^+ Amplitude

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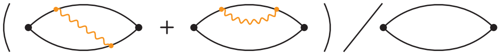

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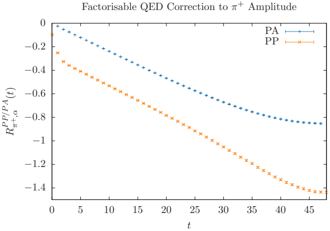
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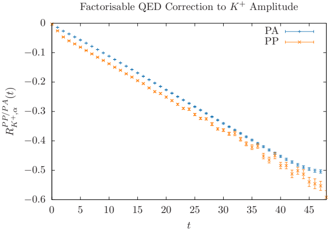
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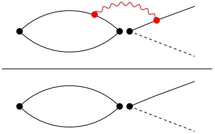
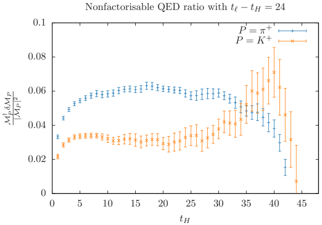


$$\longrightarrow \quad \boldsymbol{\delta^{\ell q} R}$$


Nonfactorisable QED ratio with $t_\ell - t_H = 24$

Final result $\delta R_{K,\pi}$

- Result difference $\delta R_{K\pi} = \delta R_K - \delta R_\pi$

$$\delta R_{K\pi} = -0.0086(3)_{\text{stat}}$$

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- Systematic errors

Final result $\delta R_{K,\pi}$

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$$\delta R_{K\pi} = -0.0086(3)_{\text{stat}}(^{+11}_{-4})_{\text{fit}}$$

- Systematic errors
  - varying the fit ranges for the lattice correlators

Final result $\delta R_{K,\pi}$

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$$\delta R_{K\pi} = -0.0086(3)_{\mathrm{stat}}(^{+11}_{-4})_{\mathrm{fit}}(5)_{\mathrm{disc}}$$

- Systematic errors
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  - discretisation effects $(a\Lambda_{\mathrm{QCD}})^2 \sim 5\%$

Final result $\delta R_{K,\pi}$

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$$\delta R_{K\pi} = -0.0086(3)_{\text{stat}}\left({}^{+11}_{-4}\right)_{\text{fit}}(5)_{\text{disc}}(5)_{\text{quench}}$$

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$$\delta R_{K\pi} = -0.0086(3)_{\mathrm{stat}}(^{+11}_{-4})_{\mathrm{fit}}(5)_{\mathrm{disc}}(5)_{\mathrm{quench}}(39)_{\mathrm{vol}}$$

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  - Finite volume corrections (for QED$_L$):

$$Y_{\log}(L) + Y_0 + \frac{1}{M_P L} Y_1 + \frac{1}{(M_P L)^2}\left(Y_2^{\mathrm{pt}} + Y_2^{\mathrm{sd}}\right) + \frac{1}{(M_P L)^3}\left(Y_3^{\mathrm{pt}} + Y_3^{\mathrm{sd}}\right) + \cdots$$

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$$\boxed{Y_{\log}(L) + Y_0 + \frac{1}{M_P L} Y_1} + \frac{1}{(M_P L)^2}\left(Y_2^{\rm pt} + Y_2^{\rm sd}\right) + \frac{1}{(M_P L)^3}\left(Y_3^{\rm pt} + Y_3^{\rm sd}\right) + \cdots$$

Structure independent, analytically known

[V. Lubicz *et al*, Phys. Rev. D 95, 034504 (2017)]

Final result $\delta R_{K,\pi}$

- Result difference $\delta R_{K\pi} = \delta R_K - \delta R_\pi$

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Structure independent, analytically known

[N. Tantalo *et al*, arXiv:1612.00199 ; M. Di Carlo *et al*, Phys.Rev.D 105 (2022) 7, 074509]

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Structure dependent (known form factors), analytically known

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Structure dependent, not known

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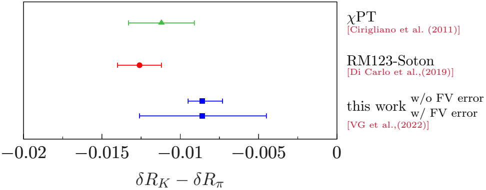
- comparison with other determinations



Determine V_{us}/V_{ud}

- Decay rate $\Gamma(P^+ \to \ell^+\nu_\ell\,[\gamma]) = \Gamma^0(P^+ \to \ell^+\nu_\ell)\,(1+\delta R_P)$

  with $\Gamma^0(P^+ \to \ell^+\nu_\ell) = \frac{G_F^2\,|V_{ij}|^2\,f_P^2}{8\pi}\,M_P\,m_\ell^2\left(1-\frac{m_\ell^2}{M_P^2}\right)^2$
- ratio of pion and kaon decay rates

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experimental input

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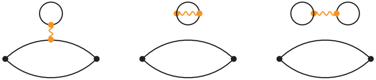
calculation

- δR_K and δR_π from our calculation
- $N_f = 2+1$ dynamical flavours, f_K/f_π from FLAG [S. Aoki *et al*, Eur.Phys.J.C 80 (2020)]

$$|V_{us}|/|V_{ud}| = 0.23154(28)_{\text{exp}}(15)_{\delta R_P}(45)_{\delta R_P,\text{vol}}(65)_{f_P}$$

Outlook - leptonic decays

- "tame" the finite volume corrections for $\delta R_{K\pi}$
  - Simulate at various different volumes
  - determine the structure dependent $1/L^3$ terms
  - use a different QED formulation
- more lattice spacings to extrapolate to continuum
- calculate sea-quark effects



- lattice calculation of real photon emission $P \to \ell\nu_\ell\gamma$
- more precise determination of decay constants f_K, f_π

- IB corrections semi-leptonic meson decays $K \to \pi\ell\nu$
  $\to$ determination of V_{us}

Summary

- Standard Model very successful, but leaves many open questions
- low-energy precision test of the Standard Model
  $\rightarrow$ QCD: first principles calculations using Monte Carlo (Lattice QCD)
- many lattice QCD calculations reaching precision of $\lesssim$ **1%**
  $\rightarrow$ need isospin breaking and QED corrections
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Thank you!

Backup

Computational Challenges

- two energy scales in the problem, box size L, lattice spacing a

$$\mathcal{O}(1/L) \ll E \ll \mathcal{O}(1/a)$$

- typical size of a lattice

$$N = L^3 \times T = 64^3 \times 128 \sim \mathcal{O}(10^7 - 10^8)$$

- Dirac-operator D: matrix of size $(12 \cdot N) \times (12 \cdot N)$

- anatomy of a lattice calculation
  - generate gauge configurations
    → need determinant $\det D$
  - "measurements": calculate quark propagators
    → need the inverse D^{-1}
    → solve the Dirac equation $D\phi = \eta$
- calculations done on supercomputers

[https://www.epcc.ed.ac.uk/facilities/dirac]

Renormalisation Scheme

- How much of IB comes from QED and how much from $m_u \neq m_d$?

$$X^{\phi} = \overline{X} + X^{SU(2)} + X^{\gamma} \qquad X^{\mathrm{QCD}} \equiv \overline{X} + X^{SU(2)}$$

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- ambiguities in choosing different schemes
- community efforts in agreeing on set of schemes
  - FLAG, g-2 Theory initiative
  - upcoming workshop in Edinburgh `https://indico.ph.ed.ac.uk/event/257/`

Photons on the Lattice

- Photon propagator $\Delta_{\mu\nu}(x-y) = \langle A_\mu(x) A_\nu(y) \rangle$ in Feynman gauge

$$\Delta_{\mu\nu}(x-y) = \delta_{\mu\nu} \frac{1}{V} \sum_{k, \vec{k} \neq 0} \frac{e^{ik(x-y)}}{k^2}$$

  V: volume of the box

- QED$_L$: remove the spatial zero-modes $\tilde{A}_\mu(k_0, \vec{k} = 0) = 0$
  [M. Hayakawa and S. Uno, Prog. Theor. Phys. 120 (2008) 413]
- other QED formulations
  - QED$_{TL}$: remove the zero-mode of the photon field, i.e. $\tilde{A}_\mu(k=0) = 0$
    [A. Duncan, E. Eichten, H. Thacker, Phys.Rev.Lett. **76**, 3894 (1996)]
  - QED$_m$: use a massive photon [M. Endres *et al.*,Phys. Rev. Lett. **117** (2016) 072002]
  - QED$_C$: C^* boundary conditions in spatial direction, i.e. fields are periodic up to charge conjugation [B. Luchini *et al.* JHEP **02** (2016) 076]
- Finite Volume corrections (depend QED formulation)

$$\text{QCD:} \quad \sim e^{-m_\pi L} \qquad\qquad \text{QED}_L\text{:} \quad \sim \frac{1}{(m_\pi L)^n}$$

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- total decay rate

$$\Gamma = \Gamma^0 + \delta\Gamma = \Gamma^0 (1 + \delta R) \qquad \delta R = \delta\Gamma / \Gamma_0$$

- with Γ^0 [PDG convention]

$$\Gamma^0 = \frac{G_F^2 \left| V_{q_1 q_2} \right|^2 (f_P^0)^2}{8\pi} \, M_P \, m_\ell^2 \left(1 - \frac{m_\ell^2}{M_P^2} \right)^2$$