Resurgence and the power of perturbation theory

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Perturbation theory is one of the most successful analytical tool to study physical processes.

Coupling constant perturbative expansions in QFT are generally asymptotic with zero radius of convergence.

Understanding the behaviour of perturbation theory at large orders is theoretically important and useful to possibly resum the series and go at strong coupling.

Resurgence appears a promising way to achieve this goal.

Aside many technicalities and some assumptions, the key principles underlying resurgence are simple.

Aim of this colloquium is to try to explain you these principles and give you a few examples of applications.

Plan

Why asymptotic series?

A brief historical excursus pre-resurgence

Basics of resurgence: the Airy function

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Conclusions

Why asymptotic series?

How can well-defined physical observables give rise to ill-defined divergent asymptotic series? Where is the ``mistake''?

In order to answer this question it is useful to consider ordinary integrals, which can be seen as toy versions of path integrals in quantum mechanics or quantum field theory. Basic ideas of resurgence are best understood in this case.

$$I(g) = \frac{1}{\sqrt{g}} \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{g}\left(\frac{x^2}{2} + \frac{x^4}{4}\right)} = \int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2} - \frac{g x^4}{4}}$$

$$= \int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2}} \sum_{n=0}^{\infty} \frac{g^n}{n!} \left(-\frac{x^4}{4}\right)^n \neq \sum_{n=0}^{\infty} \frac{g^n}{n!} \int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2}} \left(-\frac{x^4}{4}\right)^n = \sum_{n=0}^{\infty} g^n c_n$$

Well-defined and finite

Perturbation theory

Ill-defined and infinite

The series is **not** uniformly convergent and we are **not** allowed to exchange sum and integration

$$c_n \sim n! a^n \qquad (a = -4 \text{ in example})$$

If g small enough, asymptotic series provides good approximation of exact result.

They behave as convergent series if the sum is truncated to $N \lesssim \dfrac{1}{|ag|}$ terms.

After that, summing more and more terms is **not** a good idea.

The higher the coupling the less terms you should compute

Intrinsic error associated to the asymptotic series $\sim e^{-\frac{1}{ag}}$

Replace integrals with path integrals $\Longrightarrow$ Perturbative expansions in QFT

Asymptotic series do not arise only from integrals, of course.

Notably they can occur in perturbatively solving differential equations.

Extra input is needed to possibly resum the series using resummation methods.

Most useful method is so called **Borel resummation**

Divide original series by a factorial term to get a convergent series

$$\widehat{I}(t) = \sum_{n=0}^{\infty} t^n \frac{c_n}{n!} \qquad\qquad s(I) = \frac{1}{g}\int_0^{\infty} dt\, e^{-\frac{t}{g}} \widehat{I}(t)$$

$\widehat{I}(t)$, if known, can be **analytically** continued over the whole complex t-plane (Borel plane) and $s(I)$ can then be calculable

Under some (restrictive) conditions, one can prove that $s(I) = I$

This requires that $\widehat{I}(t)$ has no singularities along the positive real axis.

Example: for $c_n \sim n! a^n$ $\qquad \widehat{I}(t) = \sum_n (at)^n \sim \frac{1}{1-at}$

a<0 (alternating series) singularity for t < 0 😀

a>0 (same sign series) singularity for t>0 ☹️

A brief historical excursus pre-resurgence

Determining the large order behaviour of an asymptotic series is important to establish Borel summability of the series.

For integral/path integrals, large order behaviour around a saddle point is given by the leading coefficients of near-by saddle points!

In the example before with $f(z) = \frac{1}{2}z^2 + \frac{1}{4}z^4$, saddle point of interest is $z = 0$

Near-by saddle points are $z = \pm i$

In QFT near-by saddles are (complex) instantons.

[Vainhstein 1964; Lam 1968; Bender&Wu 1969; Lipatov 1976; …]

Before the advent of resurgence, emphasis was in finding models with Borel resummable expansions. Practically, this boiled down in determining the sign of the leading order coefficient in $n!a^n$

In special cases, like the anharmonic oscillator in QM, and 2d or 3d ϕ^4 theories, perturbation theory has been proved to be Borel resummable.

This has led to a longstanding application in statistical physics, in the study of second order phase transitions (critical phenomena).

[Baker et al., 1976; Le Guillou, Zinn-Justin, 1980; … Guida, Zinn-Justin, 1998; …]

Main observables: **critical exponents**

Two main approaches:

- epsilon expansion: start from $d = 4 - \epsilon$ and Borel resum series in ϵ for critical exponents [Wilson, Fisher, 1972]
- Fixed dimension expansion: Borel resum the $d = 3$ or $d = 2$ β-function and look for its zeros. [Parisi, 1980]

Both methods generally compatible with each other. With a few coefficient terms an accuracy $< 1\%$ is obtained.

Various algorithms developed to numerically reconstruct the Borel function from truncated series.

However, most interesting theories, including notably 4d gauge theories, are **not** Borel resummable.

Long ago it was found that if a theory contains classically dimensionless couplings, then unavoidably we have singularities in the Borel real positive axis.

[Gross&Neveu 1974; Lautrup 1977;'t Hooft 1977; …]

Such singularities are called **renormalons** and are still today little understood.

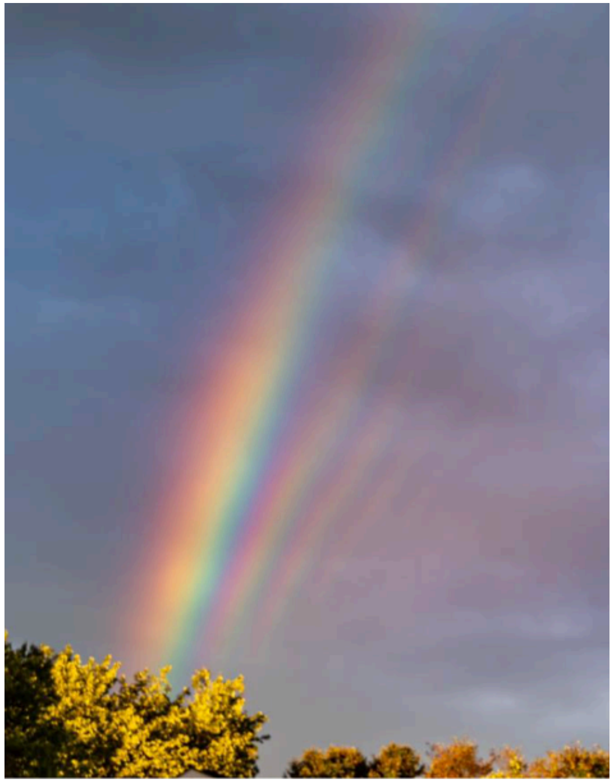
Renormalons arise from specific Feynman diagrams which give a factorially large contribution.

In contrast to instantons, where the factorial growth arises from the multiplicity of the Feynman diagrams.

What can we do if an asymptotic series is not Borel summable?

Resurgence seems to be a possible answer!

Basics of resurgence: the Airy function



$$f''(x) = xf(x)$$

Insert dummy variable ϵ and complexify $x \to z$

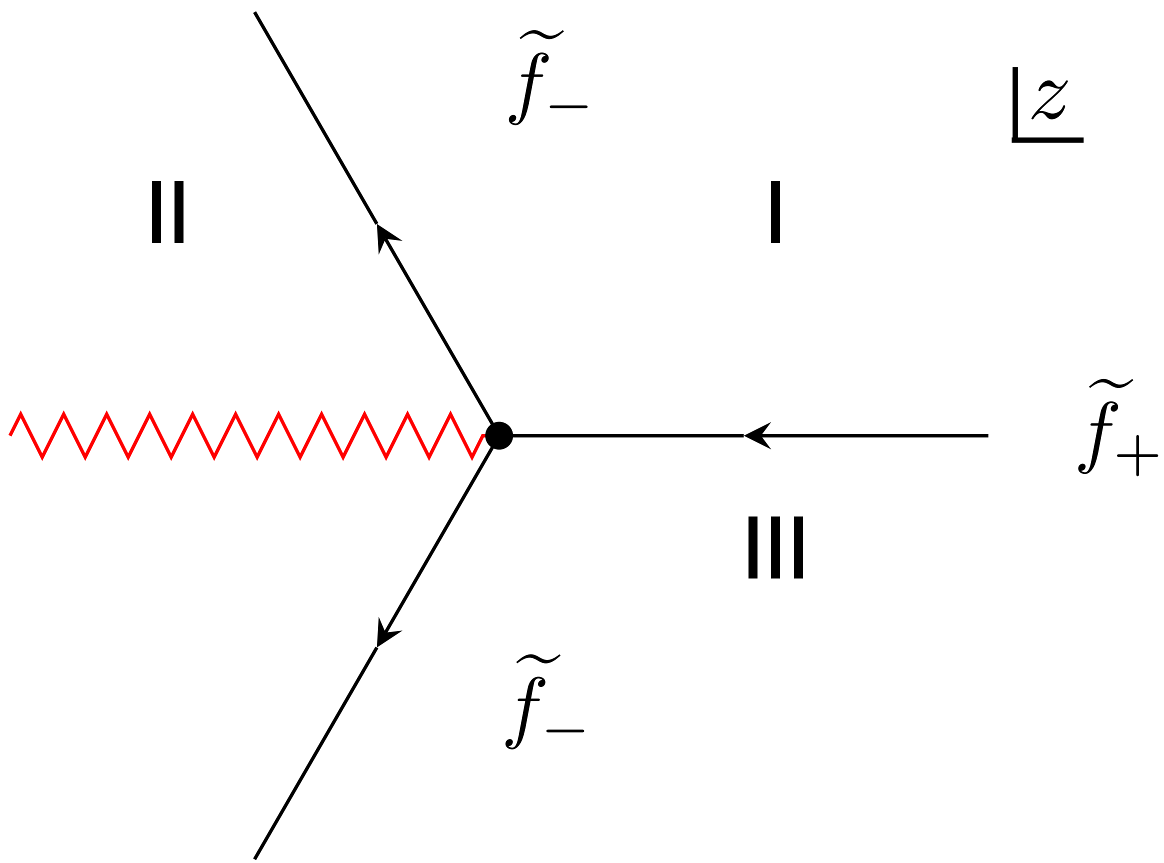
$$\epsilon f''(z) = zf(z)$$

Look for solutions $\widetilde{f}_\pm(z)$ as series in ϵ:

$$\widetilde{f}_\pm = z^{-\frac{1}{4}} e^{\pm \frac{2}{3\epsilon} z^{\frac{3}{2}}} \sum_{n=0}^{\infty} c_n^\pm (\epsilon z^{-\frac{3}{2}})^n \qquad c_n^\pm \sim n! \left(\pm \frac{3}{4} \right)^n$$

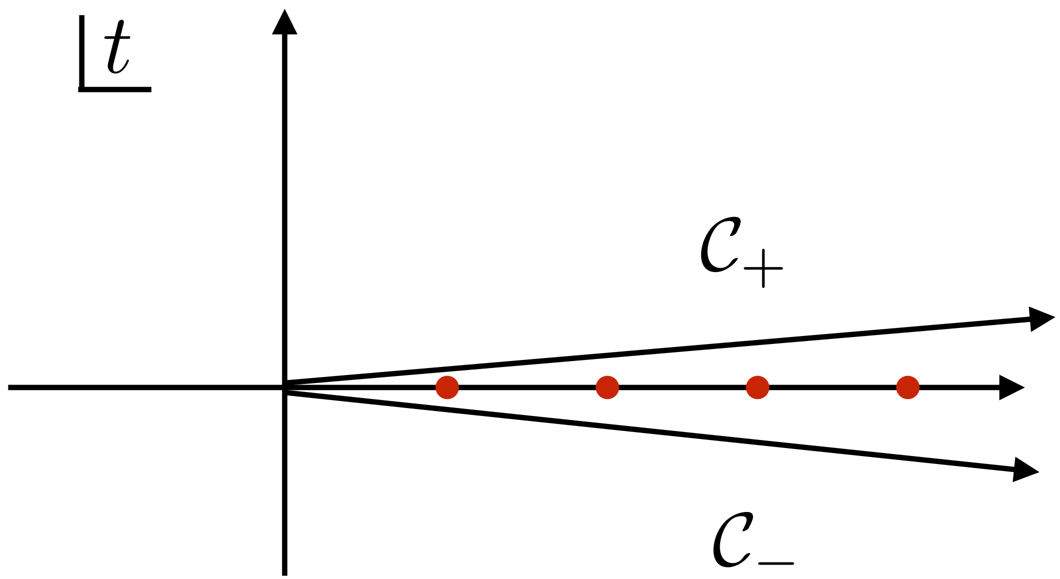
For generic z both $\widetilde{f}_{\pm}$ are Borel summable.

When $\epsilon z^{-3/2}$ is **real**, either $\widetilde{f}_{+}$ or $\widetilde{f}_{-}$ is not.



If φ is an asymptotic series,

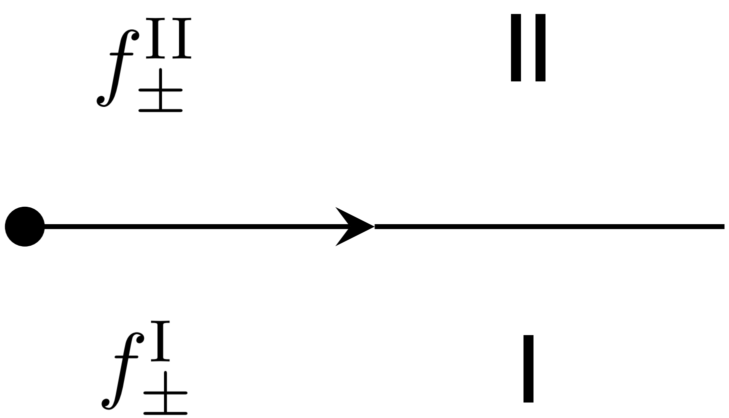
$$s_{\pm}(\varphi) = \frac{1}{\epsilon}\int_{\mathcal{C}_{\pm}} dt\, e^{-t/\epsilon}\hat{\varphi}(t)$$



$$s_+(\varphi) - s_-(\varphi) \sim e^{-1/(a\epsilon)}$$

$\boxed{\text{Stokes discontinuity}}$

First key-point: the same asymptotic series gives rise to different functions in different wedges of the complex plane because of the Stokes discontinuity



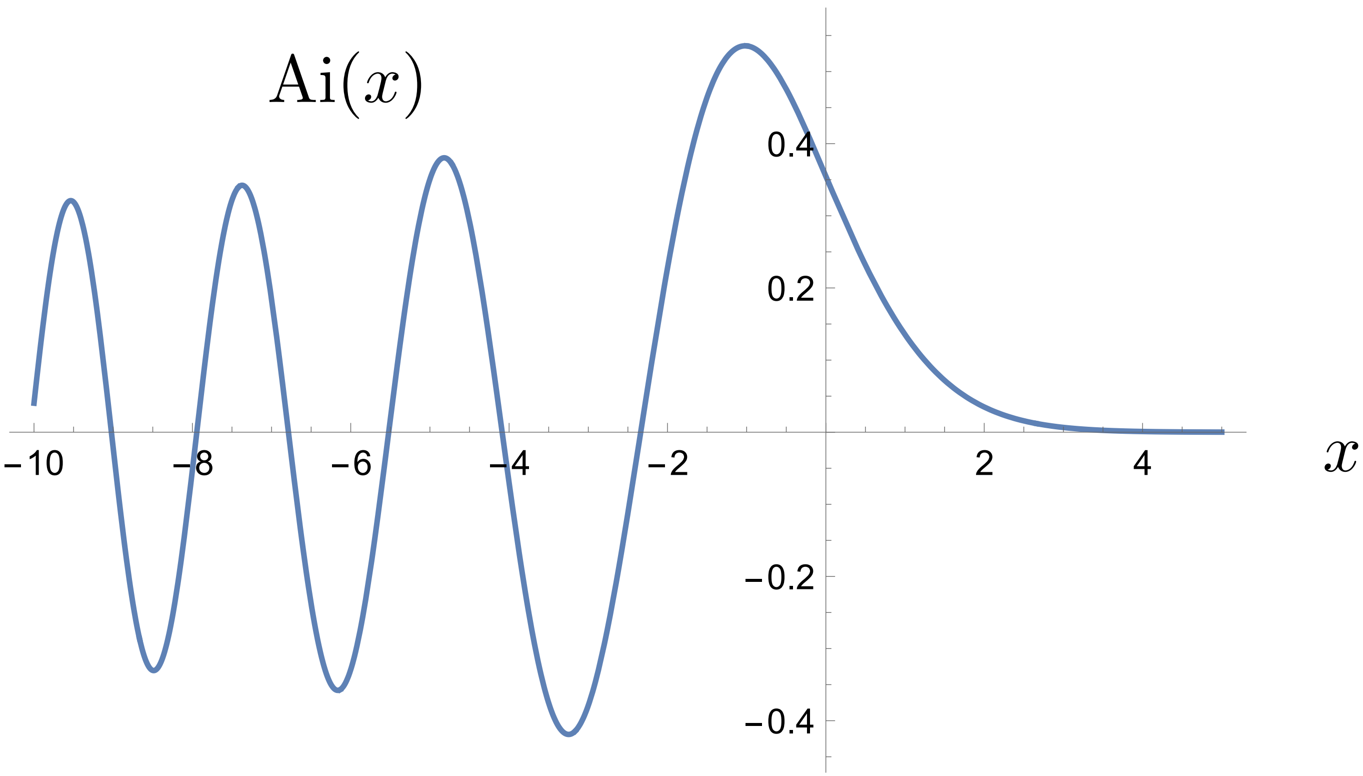
$$f_{+}^{\mathrm{II}} = f_{+}^{\mathrm{I}} \qquad\qquad f_{-}^{\mathrm{II}} = f_{-}^{\mathrm{I}} + i f_{+}^{\mathrm{I}}$$

Similarly in the other two boundaries of wedges.

Let us now go back to the original Airy function over the reals.

$$x > 0: \ \mathrm{Ai}(x) = f_{-}^{\mathrm{I}}(x)$$

$$x < 0: \ \mathrm{Ai}(x) = f_{-}^{\mathrm{I}}(x) + i f_{+}^{\mathrm{I}}(x)$$



x is the angle from main bow, resurgence in the sky!

Second key-point: thanks to the Stokes discontinuity, we can ``discover'' a second asymptotic series (f_+) from the non-Borel summability of the first (f_-)

Final observable (Ai) is in general not given by (the resummation of) a single series, but is a sum of two.

Airy function can also be written as an integral:

$$\mathrm{Ai}(z) = \int_{-\infty}^{\infty} dw\, e^{i\left(\frac{w^3}{3}+zw\right)}$$

The Stokes discontinuity is given by different **saddle dominance**.

In general, whenever a series is not Borel resummable, studying its Stokes discontinuities can allow us to get other asymptotic series which combine in the final result in a so called trans-series. In general we need an infinite number of other series.

For example $$\sum_n c_n g^n + \sum_{m,n} e^{-gm/a} c_{m,n} g^n$$

Perturbative series

Non-Perturbative transseries

When the factorial growth is given by instantons, the trans-series terms are naturally understood as the perturbative expansion around instanton configurations.

Applications in QM and QFT

QM

Quantum mechanical systems are determined by a differential equation, the Schrodinger equation.

Beautiful application of resurgence to the Schrodinger equation is at the base of the exact WKB method.

[Voros,1983; Aoki, Kawai, Takei, 1991; Delabaere, Dillinger, Pham, 1997; …]

Exact WKB is the upgrade of the WKB approximation to an exact method and allows us to get the explicit form of the trans-series for wave functions, energy eigenvalues, etc.

Trans-series terms in energy eigenvalues correspond to perturbative expansions around instantons.

In certain quantum mechanical models, the resurgence program can be successfully completed by mapping the trans-series to an appropriate (Borel resummable) single series.

[MS, Spada, Villadoro, 2016,2017]

QFT

Generally observables in QFT do not satisfy ordinary differential equations, and resurgence requires a detailed knowledge of the perturbative series to be ``activated”.

This limits its application in QFT to the few systems where we can have such knowledge.

Interesting example: known two-dimensional class of models which are

- free at high energies
- dynamically generate a mass gap at low energies
- have a continuous global symmetry which admit a large N limit

All such models are affected by **renormalon** singularities.

In fact, the existence of renormalons have historically been established in such kind of models at large N.

[Gross,Neveu, 1974; David,1982,1984; Novikov et al, 1984]

All such models are integrable.

Integrability allows us to compute many terms in perturbation theory.

[Volin, 2010]

Resurgence beautifully works!

Established both at large N (also analytically) and at finite N (numerically).

[Marino et al, Bajnok et al, 2020-2022]

First successful application of resurgence in presence of renormalon singularities.

Conclusions

Perturbation theory is one of the most successful analytical tool to study physical processes.

The generating series have often zero radius of convergence, yet this does not harm its power at weak coupling, the natural area of application of perturbation theory.

Somewhat surprisingly, perturbation theory can be used at strong coupling.

Before resurgence, this required the series to be Borel resummable, a very stringent constraint. A lot of results mostly in the context of critical phenomena.

With resurgence, we are learning how to deal with non-Borel resummable series.

The two main pillars of resurgence are:

1) Physical observables can be written as an infinite number of ``sufficiently simple" series. One of such series corresponds to the perturbative expansion, the others to series which include non-perturbative corrections.

2) Most (or all) of the information encoded in the trans-series can be reconstructed using only the perturbative series.

We expect 1) to generally apply.
2) is more subtle and requires further study.

Resurgence allows us for a systematic organization of strongly coupled theories, and an analytical understanding of non-perturbative effects, which are efficiently captured.

Resurgence requires a detailed knowledge of the perturbative series.

So far we mostly checked that resurgence works in theories where we knew the answer by other means.

Since resurgence is not a theorem in mathematics, it is useful to provide additional evidence that it works in theories where we know the answer by other means, such as 2d CFTs, non-integrable large N theories, SUSY theories, etc.

Resurgence techniques could also be used to enlarge the number of known Borel resummable theories (3d gauge theories?)

Open key question: can we use resurgence to actually compute new observables?

In quantum mechanics it has been shown that observables expressed in terms of a trans-series also admit expansions which are Borel resummable.

Generalizing that to QFT would be a way to get accurate predictions with a few terms.

Thank You