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Constructing canonical Feynman integrals with intersection theory

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based on work in collaboration with
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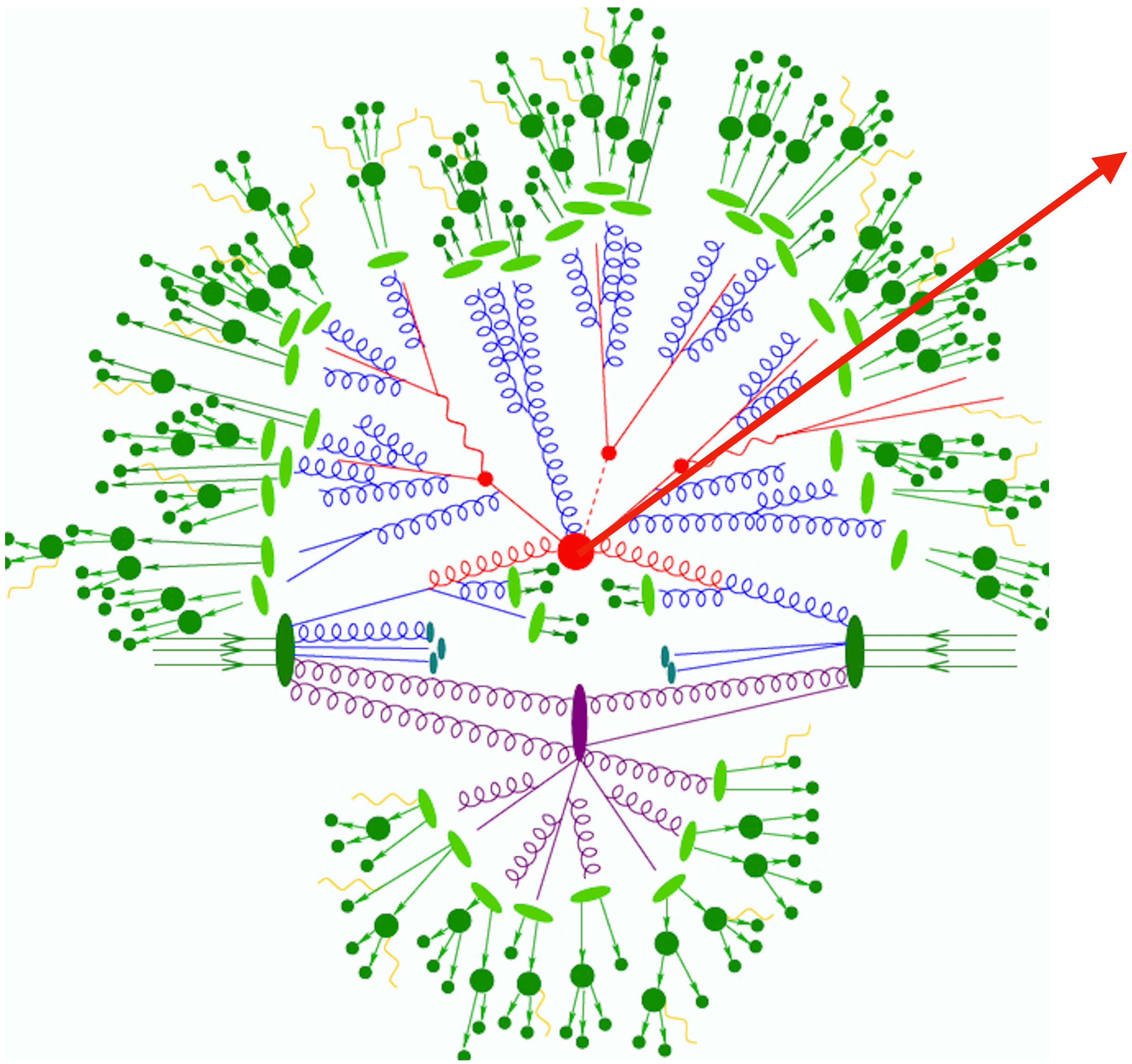
Outline

1. Introduction
2. Baikov representation and intersection theory
3. Canonical integrals for univariate case
4. Canonical integrals for general case
5. Conclusion and outlook

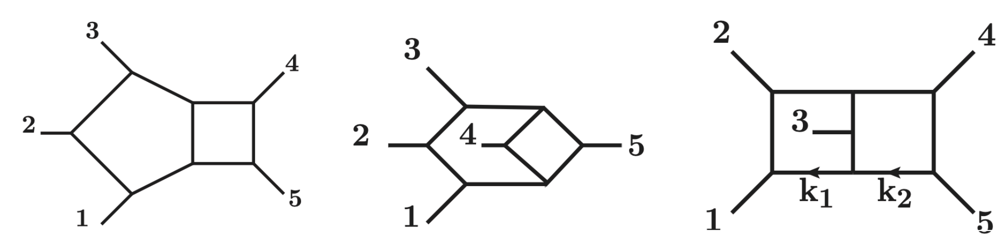
Introduction

Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia, 1812.11160
Wang, Wang, X.Xu, Xu, Yang, 2010.15649

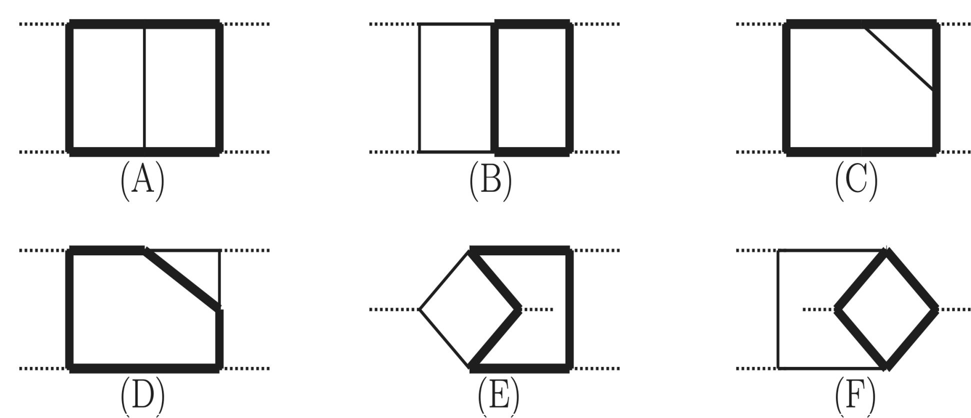
Proton-proton collision at LHC



Precise predictions of hard scattering processes require higher order corrections



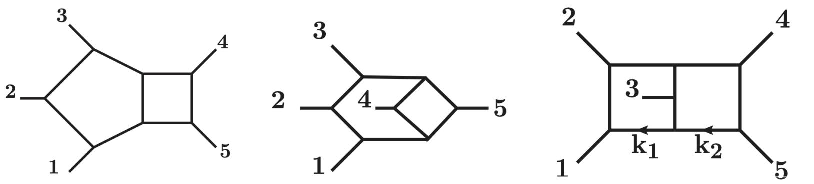
(a) penta-box | (b) hexa-box | (c) double-pentagon



Introduction

- Numerical method: sector decomposition [Heinrich '2008]

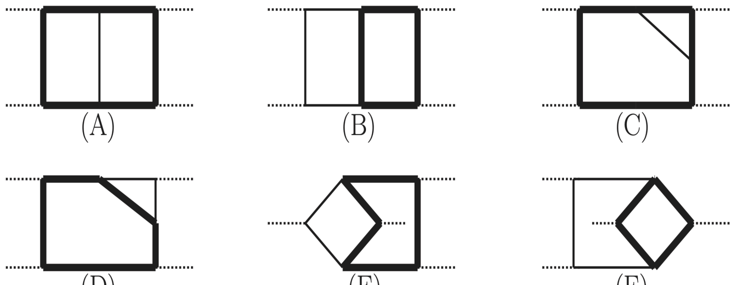
- differential equation method



(a) penta-box (b) hexa-box (c) double-pentagon

canonical master integrals [Henn '2013]

$$df_i(\epsilon, \vec{x}) = \epsilon \, dA_{ij}(\vec{x}) f_j(\epsilon, \vec{x})$$



(A) (B) (C) (D) (E) (F)

d-log form integrand

$$d\log(\alpha_1(\vec{x})) \wedge \cdots \wedge d\log(\alpha_2(\vec{x}))$$

[N.Arkani-Hamed, J.L. Bourjaily, .etc '2012]
[T.Gehrmann, J.M. Henn, T.Huber '2012]
[J.Drummond, C.Duhr, B.Eden, .etc '2013] …

Baikov representation and intersection theory

- Hypergeometric function

[K. Aomoto, M.Kita, '2011]
[Frellesvig, Gasparotto, .etc '2019]
[S.Mizera '2018]
[P.Mastrolia, S.Mizera '2019]

$$I = \int_{\mathcal{C}_R} u(\mathbf{z}) \varphi_L(\mathbf{z}) = \langle \varphi_L | \mathcal{C}_R]$$

$u(\mathbf{z}) = \prod_i p_i(\mathbf{z})^{\gamma_i}$ multiple-valued fucntion

$u(\partial \mathcal{C}) = 0$

$\varphi_L(\mathbf{z}) = \frac{q(\mathbf{z})}{\prod_i p_i(\mathbf{z})^{n_i}} dz_1 \wedge \cdots \wedge dz_n$

single valued n-form

Stoke's theorem

$$0 = \int_{\mathcal{C}_R} d(u\xi) = \int_{\mathcal{C}_R} u \left(d \log\left(u\right) \wedge + d\right) \xi$$

$$w = d\log(u) \qquad \nabla_w \equiv d + w\wedge$$

twisted cohomology $H_w^n \equiv \left\{ \langle \varphi_L | \middle| \langle \varphi_L | : \varphi_L \sim \varphi_L + \nabla_w \xi \right\}$

Baikov representation and intersection theory

- Dual integral and intersection number

[Frellesvig, Gasparotto, .etc '2019]
[Weinzierl '2020]

$$\tilde{I} = \int_{\mathcal{C}_L} u(\mathbf{z})^{-1} \varphi_R(\mathbf{z}) = [\mathcal{C}_L|\varphi_R\rangle$$

twisted cohomology $H^n_{-w} \equiv \left\{ |\varphi_R\rangle \middle| |\varphi_R\rangle : \varphi_R \sim \varphi_R + \nabla_{-w}\xi \right\}$

intersection number

$$\langle\varphi_L|\varphi_R\rangle = \frac{1}{(2\pi i)^n} \int \varphi_L^c \varphi_R$$

φ_L^c is the same cohomology but vanished on boundary

computation of intersection number

univariate case

$$\langle\varphi_L|\varphi_R\rangle = \sum_{p \in \text{pole of } w} \mathbf{Res}_{z=p}\left(\psi\varphi_R\right)$$

$$\nabla_w \psi = \varphi_L$$

Baikov representation and intersection theory

- Reduction

$$I = \int_{\mathcal{C}} u(\mathbf{z}) \varphi_L(\mathbf{z})$$

↓

master integrals $\quad J_i = \int_{\mathcal{C}} u(\mathbf{z}) e_i(\mathbf{z})$

$$I = \sum_{i=1}^{\nu} c_i J_i \quad \text{or} \quad \langle \varphi_L | = \sum_{i=1}^{\nu} c_i \langle e_i |$$

$$\nu = \{\text{the number of solutions of } w = 0\}$$

↓

$$c_i = \sum_j \langle \varphi | h_j \rangle (\mathbf{C}^{-1})_{ji}\,, \quad \mathbf{C}_{ij} = \langle e_i | h_j \rangle\,.$$

dual basis $\quad \{|h_i\rangle\} \in H^n_{-w}$

Baikov representation and intersection theory

- Baikov representation

L-loop Feynman integral with E+1 external legs

$$F_{a_1,\ldots,a_N} = \int \left[\prod_{i=1}^{L} \frac{d^d k_i}{i\pi^{d/2}} \right] \frac{1}{D_1^{a_1}\, D_2^{a_2} \cdots D_N^{a_N}} \qquad N = \frac{L(L+1)}{2} + LE$$

standard Baikov representation

$$F_{a_1,\cdots,a_N} = CG(p_1,\ldots,p_E)^{\frac{E-D+1}{2}} \int_{\mathcal{C}} G(k_1,\ldots,k_L,p_1,\ldots,p_E)^{\frac{D-L-E-1}{2}} \frac{dz_1\cdots dz_N}{z_1^{a_1}\cdots z_N^{a_N}}$$

$G(\{q_i\}) = \det(q_i \cdot q_j)$ and $z_i \equiv D_i$

loop-by-loop Baikov representation

$$F_{a_1,\ldots,a_N} = \mathcal{N}_\epsilon \int_{\mathcal{C}} \left[\prod_i \left[G_i(\mathbf{z})\right]^{-\gamma_i-\beta_i\epsilon} \right] \prod_{j=1}^{n} \frac{dz_j}{z_j^{\alpha_j}} \qquad \mathbf{z} \equiv \{z_1,\ldots,z_n\}$$

$\downarrow$ $u(\mathbf{z})$ $\qquad$ $\downarrow$ $\varphi_L(\mathbf{z})$

Baikov representation and intersection theory

- d-log form integral

[Henn ‘2013]

canonical differential equations

$$d\vec{F}(\epsilon,\mathbf{x}) = \epsilon dA(\mathbf{x})\vec{F}(\epsilon,\mathbf{x}) = \epsilon \sum_i d\log(\alpha_i(\mathbf{x}))\vec{F}(\epsilon,\mathbf{x})$$

$$\vec{F}(\epsilon,\mathbf{x}) = \mathbf{P}\exp\left[\epsilon \sum_i \int_\gamma d\log(\alpha_i(\mathbf{x}))\right]\vec{F}(\epsilon,x_0)$$

canonical integrals in Baikov representation

$$\int_{\mathcal{C}} \left[\prod_i \left[G_i(\mathbf{z})\right]^{-\beta_i \epsilon}\right] \prod_{j=1}^{n} d\log f_j(\mathbf{z})$$

Questions:

1. How to construct d-log form integrand?
2. How to transform it to Feynman integrals?

Canonical integrals for univariate case

- univariate case

$$u(z) = \prod_i G_i(z)^{-\gamma_i - \beta_i \epsilon}$$

all γ_i are integer

$$u(z) = \frac{\mathcal{K}_1^\epsilon}{\mathcal{K}_0} \prod_{j=0}^{\nu} (z - c_j)^{-\gamma'_j - \beta'_j \epsilon}$$

$$\hat{\phi}_i(z) = \frac{\mathcal{K}_0}{z - c_i} \prod_{j=0}^{\nu} (z - c_j)^{\gamma'_j}\,,\ (i = 1, \ldots, \nu)$$

one half-integer γ_i and two distinct roots

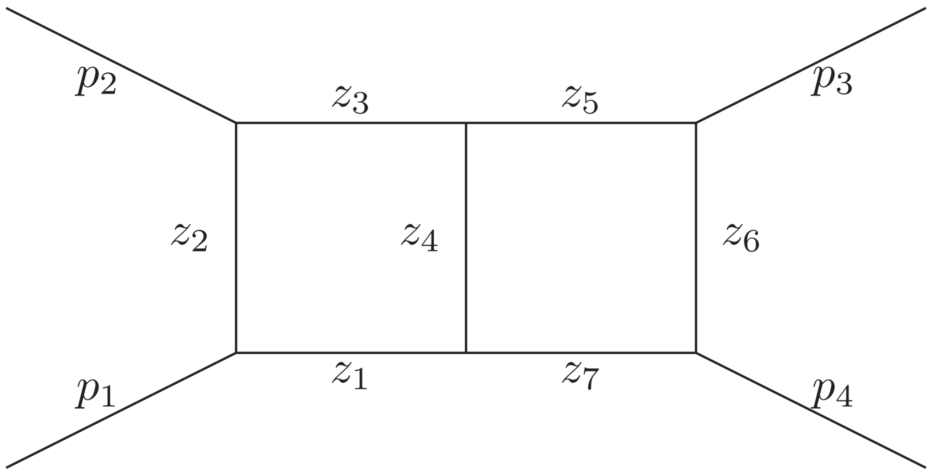
$$u(z) = \frac{\mathcal{K}_1^\epsilon}{\mathcal{K}_0} \big[(z - c_0)(z - c_1)\big]^{-\gamma_1 - \beta_1 \epsilon} \prod_{j=2}^{\nu} (z - c_j)^{-\gamma'_j - \beta'_j \epsilon}$$

$$\hat{\phi}_1(z) = \frac{\mathcal{K}_0}{\big[(z - c_0)(z - c_1)\big]^{1/2 - \gamma_1}} \prod_{j=2}^{\nu} (z - c_j)^{\gamma'_j}$$

$$\hat{\phi}_i(z) = \frac{\mathcal{K}_0}{z - c_i} \frac{\sqrt{(c_0 - c_i)(c_1 - c_i)}}{\big[(z - c_0)(z - c_1)\big]^{1/2 - \gamma_1}} \prod_{j=2}^{\nu} (z - c_j)^{\gamma'_j}\,,$$

Canonical integrals for univariate case

- Maximal-cut double box



$$u(z) = \frac{1}{s^2}\left(\frac{t(s+t)}{s^2}\right)^{\epsilon} z^{-1-\epsilon}(s+z)^{\epsilon}(t-z)^{-1-2\epsilon}$$

Master integrals

$$E_1 = F_{1,1,1,1,1,1,1,0,0} \to \langle e_1| = dz$$

$$E_2 = F_{1,2,1,1,1,1,1,0,0} \to \langle e_2| = \frac{1+2\epsilon}{z}dz$$

canonical integrals

$$I_1 \to \phi_1 = s^2 z dz$$

$$I_2 \to \phi_2 = s^2(t-z)dz$$

Canonical integrals for univariate case

- reduction

$$I_1 = -\frac{s(1+3\epsilon)}{2\epsilon} E_1 + \frac{st(1+\epsilon)}{2\epsilon(1+2\epsilon)} E_2$$
$$I_2 = \frac{s(1+3\epsilon)+2\epsilon t}{2\epsilon} E_1 - \frac{st(1+\epsilon)}{2\epsilon(1+2\epsilon)} E_2$$

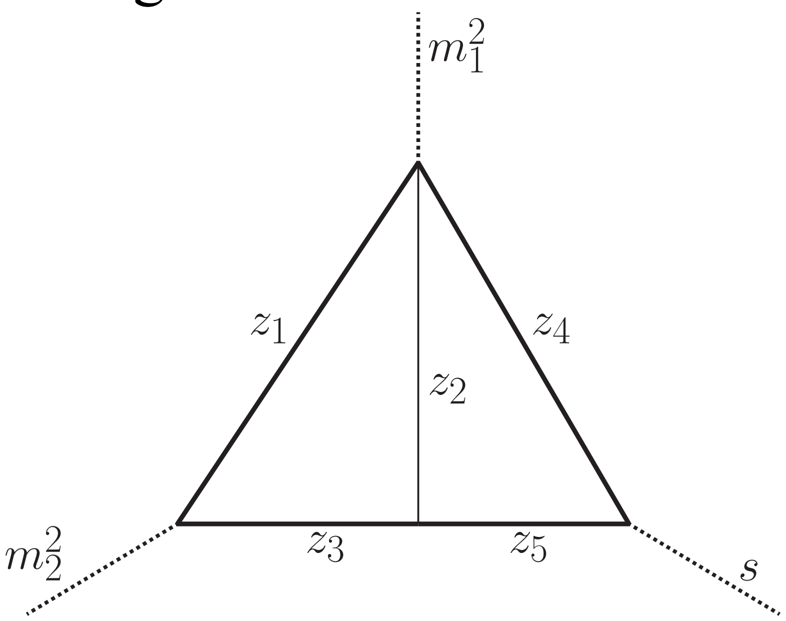
- differential equations

$$\frac{\partial}{\partial s}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \epsilon \begin{pmatrix} -\frac{2}{s} & \frac{1}{s+t} \\ \frac{2}{s} & -\frac{s+2t}{s(s+t)} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

$$\frac{\partial}{\partial t}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \epsilon \begin{pmatrix} 0 & -\frac{s}{t(s+t)} \\ -\frac{2}{t} & -\frac{s}{t(s+t)} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

Canonical integrals for univariate case

- Maximal-cut two-loop triangle



$$u(z) = \frac{1}{\sqrt{-\lambda}}\, z^{-2\epsilon}\left[-(z-c_0)(z-c_1)\right]^{-1/2+\epsilon}\left[(z-c_2)(z-c_3)\right]^{-\epsilon}$$

master integrals

$$F_{1,1,1,1,1,0,0}$$
$$F_{2,1,1,1,1,0,0}$$
$$F_{2,1,1,2,1,0,0}$$
$$F_{1,1,1,2,1,0,0}$$

canonical integrals

$$\hat{\phi}_1(z) = \sqrt{\lambda}\,, \quad \hat{\phi}_4(z) = \sqrt{\lambda}\frac{\sqrt{c_0 c_1}}{z}\,,$$

$$\hat{\phi}_{2,3}(z) = \sqrt{\lambda}\frac{\sqrt{(c_0 - c_{2,3})(c_1 - c_{2,3})}}{z - c_{2,3}}$$

Canonical integrals for general case

- general case

$$u(\mathbf{z}) = \prod_i \left[G_i(\mathbf{z})\right]^{-\gamma_i - \beta_i \epsilon}$$

↓ pick up one of the variables $u(\mathbf{z}) = u_1(\mathbf{z}) u'(\mathbf{z}')$

Construct d-log for z_1: $u(\mathbf{z})\hat{\varphi}_i^{(1)}(\mathbf{z}) dz_1$

↓ involving $\sqrt{(c_0 - c_i)(c_1 - c_i)}$

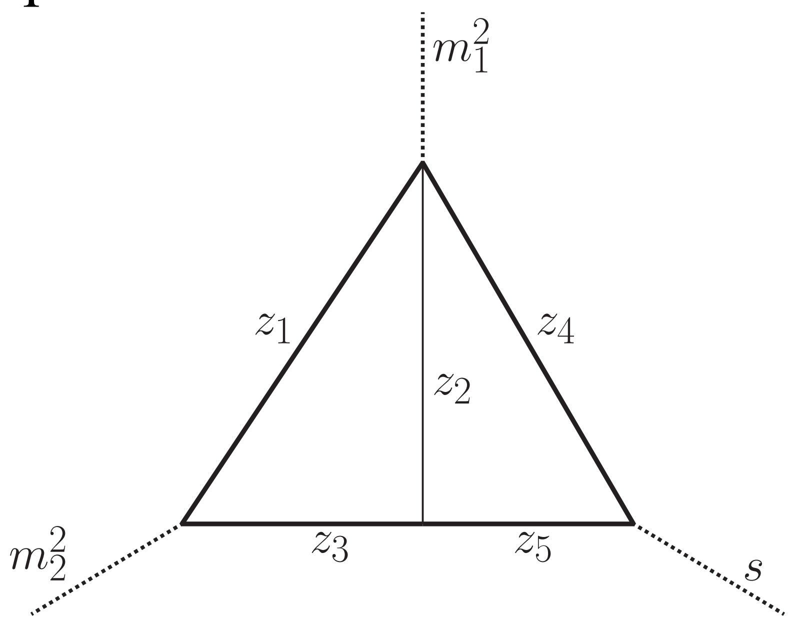
Linear combination of $\hat{\varphi}_i^{(1)}(\mathbf{z})$ making it rational or $\sqrt{\Lambda(\mathbf{z}')}\, g(\mathbf{z})$

↓ u for remaining variables $\dfrac{u'(\mathbf{z}')}{\sqrt{\Lambda(\mathbf{z}')}}$

Iterated constructing d-log for remaining variables

Canonical integrals for general case

- two-loop triangle example



$$u(\mathbf{z}) = \frac{(-\lambda)^{-1/2}\, s^{-\epsilon}(z_3+m^2)^{-\epsilon}}{\left[-(z-c_0)(z-c_1)\right]^{1/2-\epsilon}} \prod_{i=2}^{5}(z-c_i)^{-\epsilon}$$

roots

$$c_{0,1} = m_2^2 + z_5 \pm 2\sqrt{m_2^2(m^2+z_5)}\,,$$

$$c_{2,3} = \frac{1}{2s}\Big[z_4(s-m_1^2+m_2^2) + z_5(s+m_1^2-m_2^2) + s(m_1^2+m_2^2-s) \pm \sqrt{\lambda\,\rho_1(z_4,z_5)}\Big]$$

Canonical integrals for general case

- d-log form associated with c_2 and c_3

$$\hat{\phi}_2^{(1)}(\mathbf{z}) = \sqrt{\lambda}\frac{\sqrt{(c_0-c_2)(c_1-c_2)}}{z-c_2} = \sqrt{\lambda}\frac{\sqrt{\lambda}(s-z_4+z_5)-\sqrt{\rho_1}(s-m_1^2+m_2^2)}{2s(z-c_2)},$$
$$\hat{\phi}_3^{(1)}(\mathbf{z}) = \sqrt{\lambda}\frac{\sqrt{(c_0-c_3)(c_1-c_3)}}{z-c_3} = \sqrt{\lambda}\frac{\sqrt{\lambda}(s-z_4+z_5)+\sqrt{\rho_1}(s-m_1^2+m_2^2)}{2s(z-c_3)}.$$

- linear combination

$$\hat{\varphi}_2^{(1)}(\mathbf{z}) = \hat{\phi}_2^{(1)}(\mathbf{z}) + \hat{\phi}_3^{(1)}(\mathbf{z}) = \frac{2\lambda(s-z_4+z_5)}{2s(z-c_2)}$$
$$\hat{\varphi}_3^{(1)}(\mathbf{z}) = \hat{\phi}_3^{(1)}(\mathbf{z}) - \hat{\phi}_2^{(1)}(\mathbf{z}) = \frac{2\sqrt{\lambda}\sqrt{\rho_1}(s-m_1^2+m_2^2)}{2s(z-c_3)}$$

- construct d-log form for remaining variables

$$\hat{\varphi}_2(\mathbf{z}) = \frac{2\lambda(s-z_4+z_5)}{2s(z-c_2)}\frac{1}{z_1z_2z_3z_4z_5}$$
$$\hat{\varphi}_3(\mathbf{z}) = \frac{2\sqrt{\lambda}\sqrt{\rho_1}(s-m_1^2+m_2^2)}{2s(z-c_3)}\frac{\sqrt{s(s-4m^2)}}{\sqrt{\rho_1}z_1z_2z_3z_4z_5}$$

Canonical integrals for general case

- reduction to master integrals

$$\begin{aligned}
\langle\varphi_2| &= \frac{s(s-m_1^2-m_2^2)}{\epsilon}\langle F_{11121}| - \frac{m_2^2(s+m_1^2-m_2^2)}{\epsilon}\langle F_{21111}| \\
&+ \frac{m^2\lambda + s m_1^2 m_2^2}{\epsilon^2}\langle F_{21121}| + \frac{2\big[2m^2(s-m_1^2+m_2^2)-sm_2^2\big]}{\epsilon^2}\langle F_{10221}| \\
&+ \frac{m_2^2}{\epsilon^2}\langle F_{21002}| + \frac{s}{\epsilon^2}\langle F_{01220}| - \frac{m_1^2}{\epsilon^2}\langle F_{21020}|\,, \\
\langle\varphi_3| &= \frac{\sqrt{\lambda}\sqrt{s(s-4m^2)}}{\epsilon}\langle F_{11121}|\,.
\end{aligned}$$

Summary and outlook

Summary

1. introduce basis idea of intersection theory and reduction

2. How to construct d-log form integrand in Baikov representation

3. Transform the d-log form integrand into Feynman integrals with intersection theory

Outlook

1. Apply to some other examples

2. Integrand beyond d-log form for elliptic integrals

3.Geometric properties of Feynman integrals