Two Stories About Axion Physics

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June 15, 2021 online seminar at Mainz

based on:
2104.03267 with Buen-Abad, Fan, Sun;
2105.09950 with Fan, Fraser, Stout

What is an axion?

In my usage, "axion" = **"axion-like particle"**, not necessarily the QCD axion.

The axion is a **periodic** scalar field, $a \cong a + 2\pi f_a$, with f_a being the decay constant.

With *approximate* continuous shift symmetry. Two ways such a field can arise:

- As a (pseudo-)Nambu-Goldstone boson of a spontaneously broken U(1) global symmetry.

- As a Wilson loop of a higher-dimensional U(1) gauge field, like $a(x) = \oint \mathrm{d}x^5 A_5(\vec{x}, x^5)$, where the periodicity is inherited from A_5 gauge transformations with winding.

  (Could also be higher-form generalizations, $\oint \mathrm{d}x^5 \mathrm{d}x^6 B_{56}(\vec{x}, x^5, x^6)$, etc.)

Part 1. Axions and the $g-2$ Anomaly

arXiv:2104.03267 with Buen-Abad, Fan, Sun

Axion Basics

Axion arises ubiquitously from top-down theory (Svrcek, Witten 2006) and low-energy phenomenology theories (Peccei, Quinn, Wilczek, Weinberg .....).

Exact discrete **shift symmetry** on the field and approximate continuous shift symmetry in the axion's interactions. For example,

$$\frac{\partial_\mu a}{f_a}\left(\bar{\ell}_L k_L \gamma^\mu \ell_L + \bar{\ell}_R k_R \gamma^\mu \ell_R\right) \qquad \ell : \text{SM charged leptons}$$

k_L, k_R are 3×3 Hermitian matrices in the mass basis. The off-diagonal entries could be severely constrained by lepton flavor violation processes (more later). First we consider **flavor-diagonal** entries.

Low-energy axion EFT for muon g-2

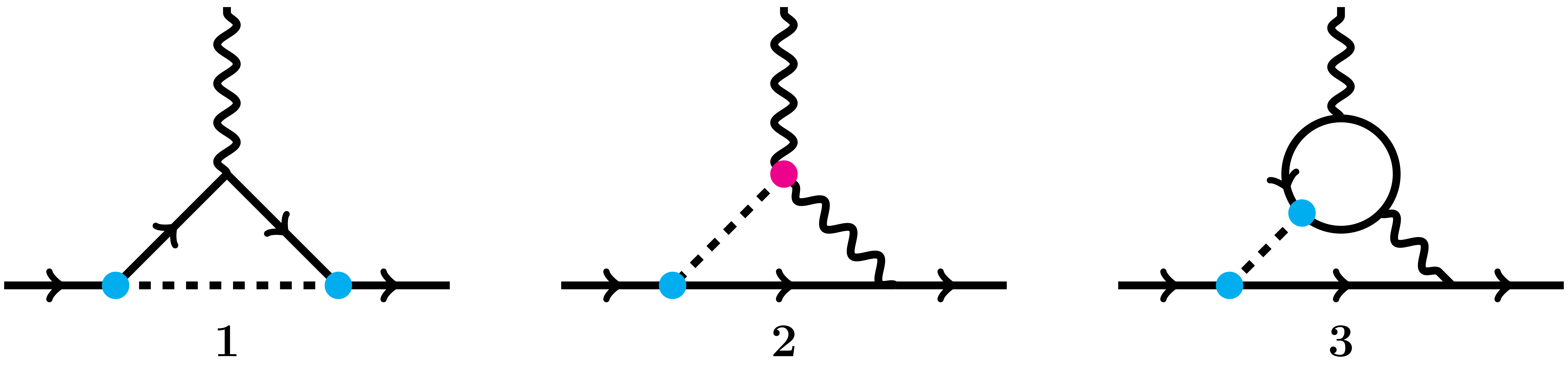
$$\mathscr{L}_{\rm eff} \supset \frac{c_{ii}}{2}\frac{\partial_\mu a}{f_a}\left(\bar{\ell}_i \gamma^\mu \gamma^5 \ell_i\right) + c_{\gamma\gamma}\frac{\alpha}{4\pi}\frac{a}{f_a}F_{\mu\nu}\tilde{F}^{\mu\nu} + \cdots$$

$c_{ii} = (k_R)_{ii} - (k_L)_{ii}$ in lepton mass basis

Quantized "Chern-Simons term"
e^{iS} is invariant only if $c_{\gamma\gamma}$ is quantized and takes rational value (canonically normalized field strength)

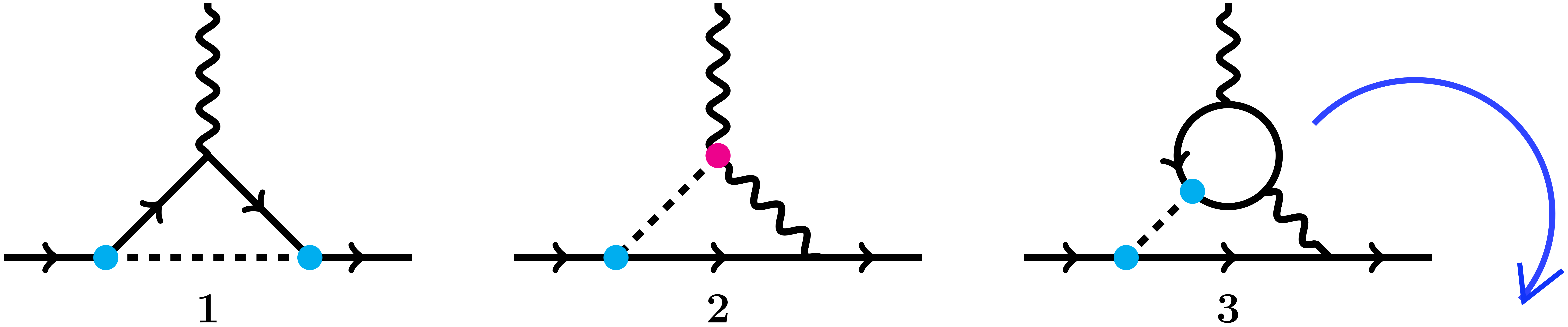
Side comment: one could encounter an irrational $c_{\gamma\gamma}$. For example, the infrared contribution to the leading order coupling of QCD axion to the photons through mixing with the pions is irrational. But **no periodicity violation!** The reason is that the leading order coupling and high-order couplings (e.g: $a^2F\tilde{F}, a^3F\tilde{F}$) could be repacked into a periodic function. (Appendix A of 1709.06085 Agrawal, Fan, Reece and Wang).

$$\mathscr{L}_{\text{eff}} \supset \frac{c_{ii}}{2}\frac{\partial_\mu a}{f_a}\left(\bar{\ell}_i\gamma^\mu\gamma^5\ell_i\right) + c_{\gamma\gamma}\frac{\alpha}{4\pi}\frac{a}{f_a}F_{\mu\nu}\tilde{F}^{\mu\nu} + \cdots$$



Chang et.al 2001; Marciano et.al. 2016; Bauer et.al. 2017.

$$\mathscr{L}_{\text{eff}} \supset \frac{c_{ii}}{2}\frac{\partial_\mu a}{f_a}\left(\bar{\ell}_i\gamma^\mu\gamma^5\ell_i\right) + c_{\gamma\gamma}\frac{\alpha}{4\pi}\frac{a}{f_a}F_{\mu\nu}\tilde{F}^{\mu\nu} + \cdots$$

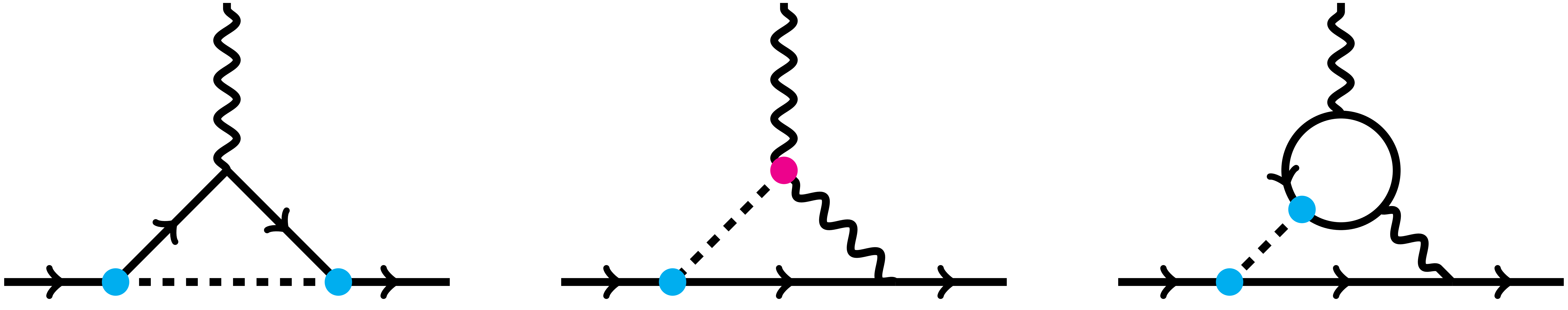


light fermion loop could not be reduced to a contact interaction.

Buen-Abad, Fan, Reece and Sun 2021

Chang et.al 2001; Marciano et.al. 2016; Bauer et.al. 2017.

$$\mathscr{L}_{\text{eff}} \supset \frac{c_{ii}}{2}\frac{\partial_\mu a}{f_a}\left(\bar{\ell}_i\gamma^\mu\gamma^5\ell_i\right) + c_{\gamma\gamma}\frac{\alpha}{4\pi}\frac{a}{f_a}F_{\mu\nu}\tilde{F}^{\mu\nu} + \cdots$$



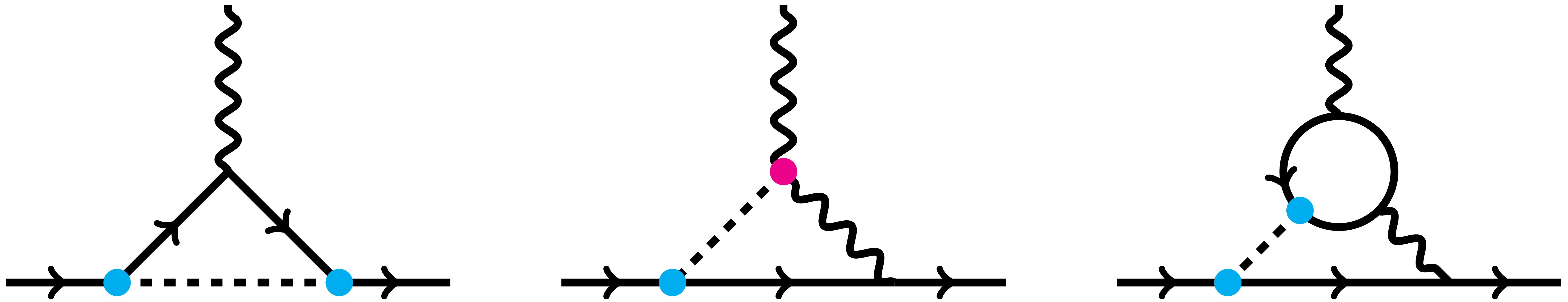
1 2 3

$$\Delta a_\mu^{(1)} \propto -\frac{c_{\mu\mu}^2}{16\pi^2}, \quad \Delta a_\mu^{(2)} \propto -\frac{c_{\mu\mu}c_{\gamma\gamma}\alpha}{16\pi^3}, \quad \Delta a_\mu^{(3)} \propto -\frac{c_{\mu\mu}c_{ii}\alpha}{16\pi^3},$$

Chang et.al 2001; Marciano et.al. 2016; Bauer et.al. 2017.

Different combinations of the axion couplings: Darme et.al 2020

Case 1: only $c_{\mu\mu}$, $c_{\gamma\gamma}$



1 2 3

$$\Delta a_\mu^{(1)} \propto -\frac{c_{\mu\mu}^2}{16\pi^2}, \quad \Delta a_\mu^{(2)} \propto -\frac{c_{\mu\mu} c_{\gamma\gamma}\alpha}{16\pi^3} \quad \Delta a_\mu^{(3)} \propto -\frac{c_{\mu\mu}^2 \alpha}{16\pi^3}$$

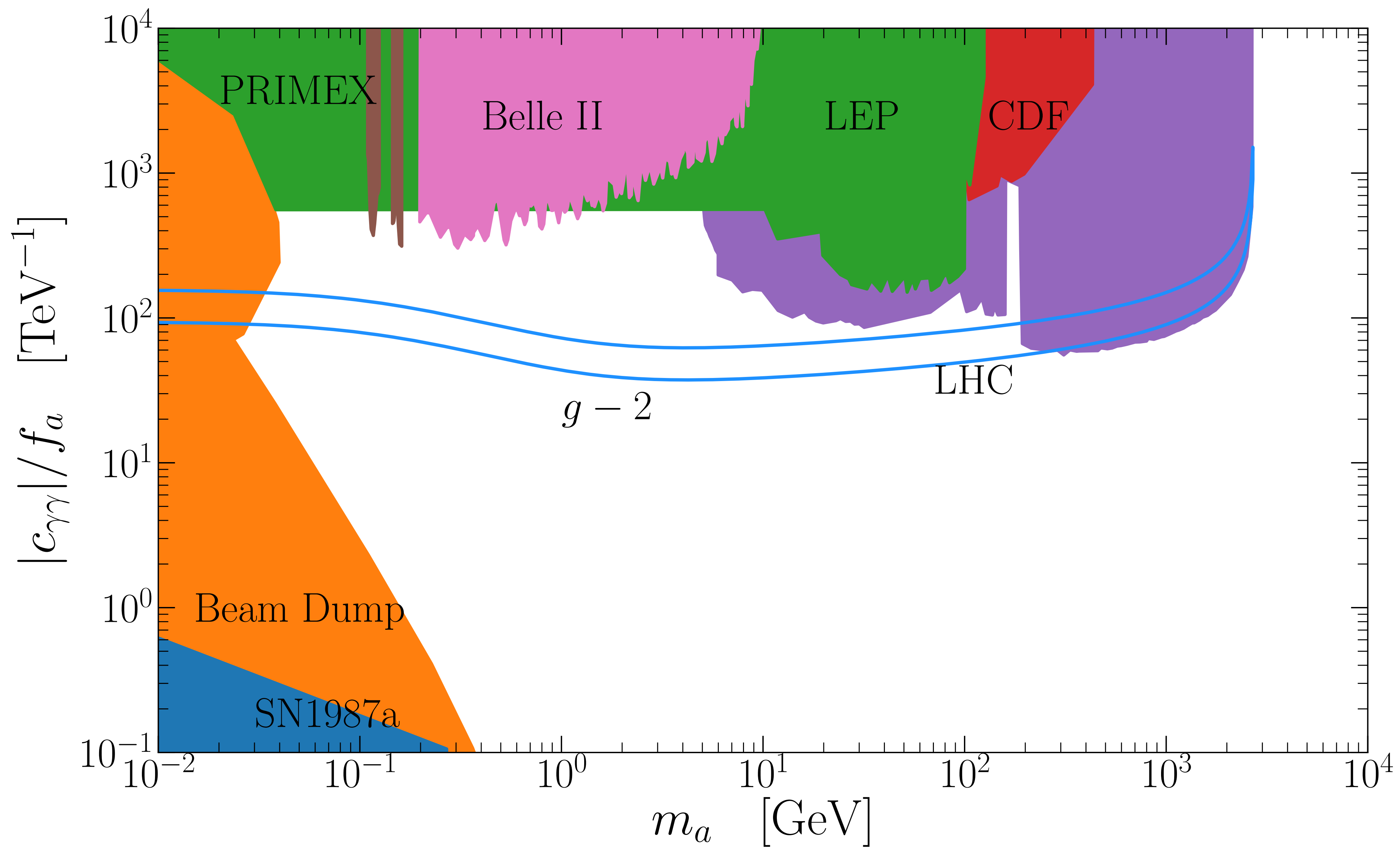
$$\Rightarrow c_{\mu\mu}/c_{\gamma\gamma} < 0$$

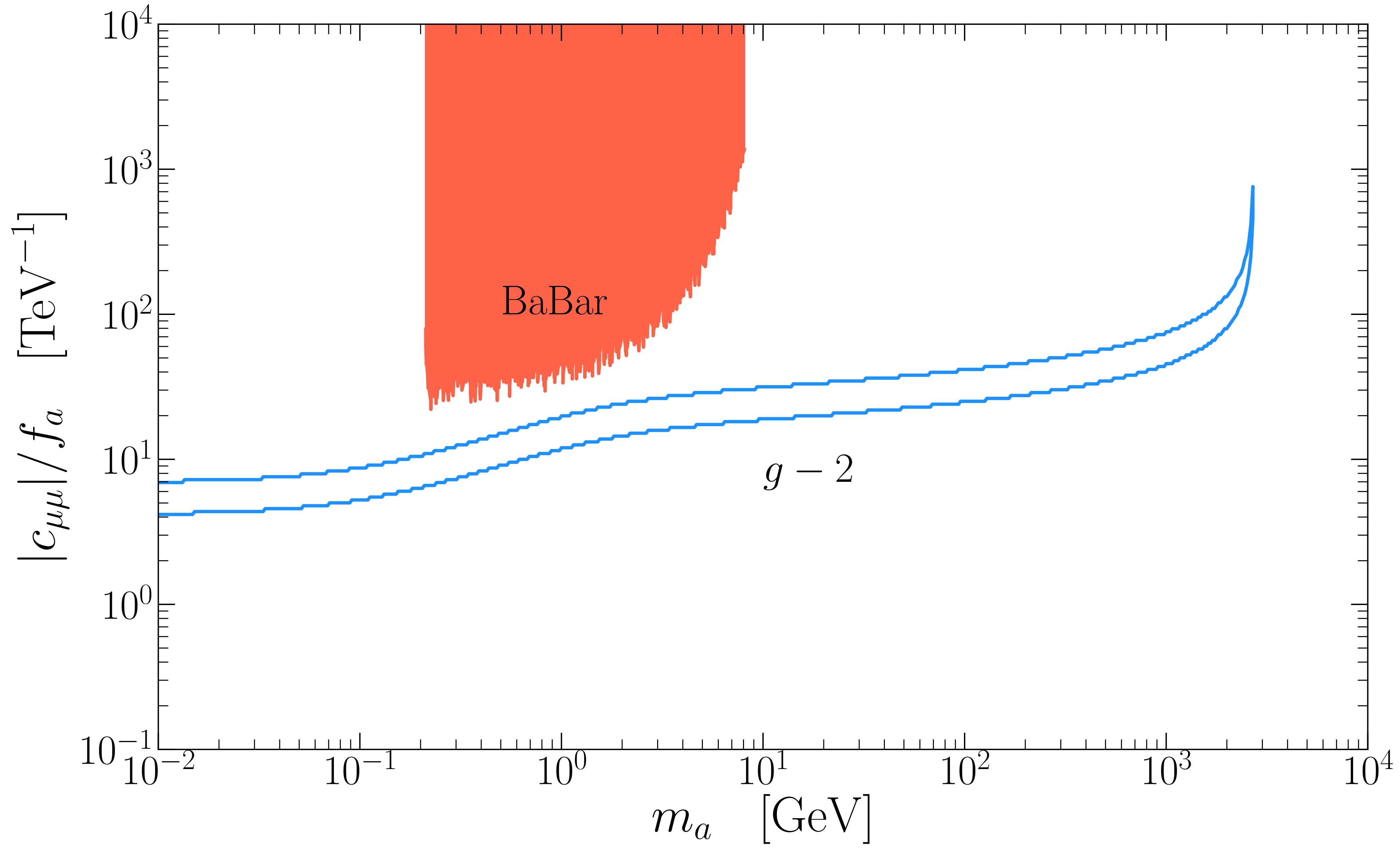
Case 1: only $c_{\mu\mu}$, $c_{\gamma\gamma}$

$$c_{\gamma\gamma}/c_{\mu\mu} < 0, \quad m_a \subset (40\,\mathrm{MeV} - 200\,\mathrm{GeV})$$

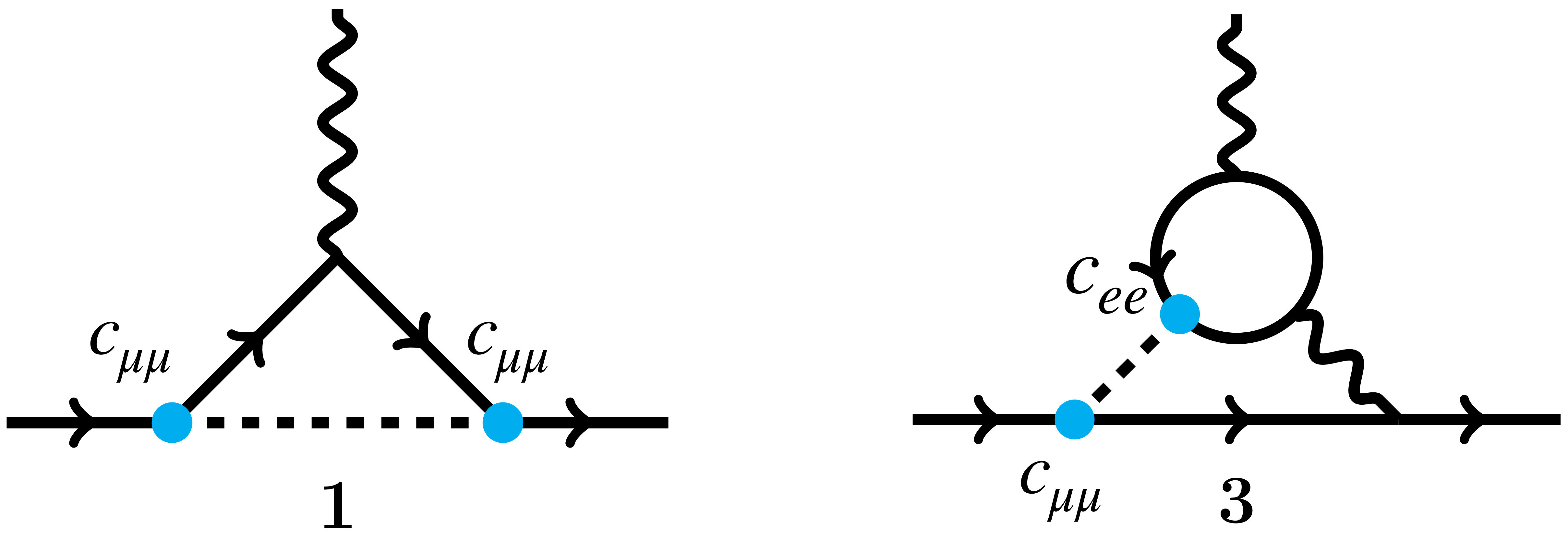
$$\left|\frac{f_a}{c_{\gamma\gamma}}\right| \lesssim (10-25)\,\mathrm{GeV}, \quad \left|\frac{f_a}{c_{\mu\mu}}\right| \lesssim 100\,\mathrm{GeV}\ .$$

Buen-Abad, Fan, Reece, Sun, 2021





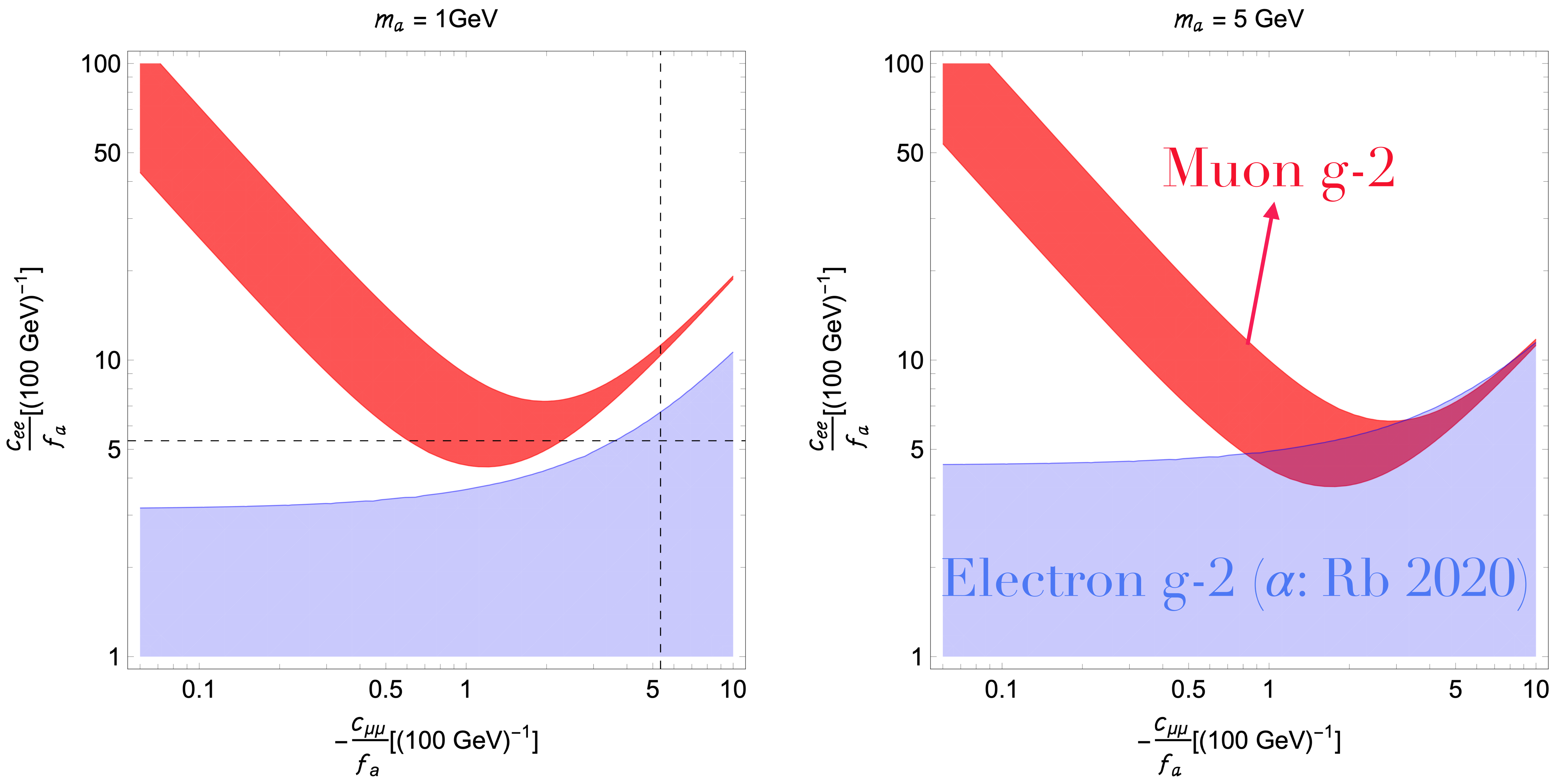
Case 2: only $c_{\mu\mu}$, c_{ee}



$$\Delta a_\mu^{(1)} \propto -\frac{c_{\mu\mu}^2}{16\pi^2}, \quad \Delta a_\mu^{(3)} \propto -\frac{c_{\mu\mu} c_{ee} \alpha}{16\pi^3}$$

$$\Rightarrow c_{\mu\mu}/c_{ee} < 0$$

Case 2: only $c_{\mu\mu}$, c_{ee}



Buen-Abad, Fan, Reece, Sun, 2021

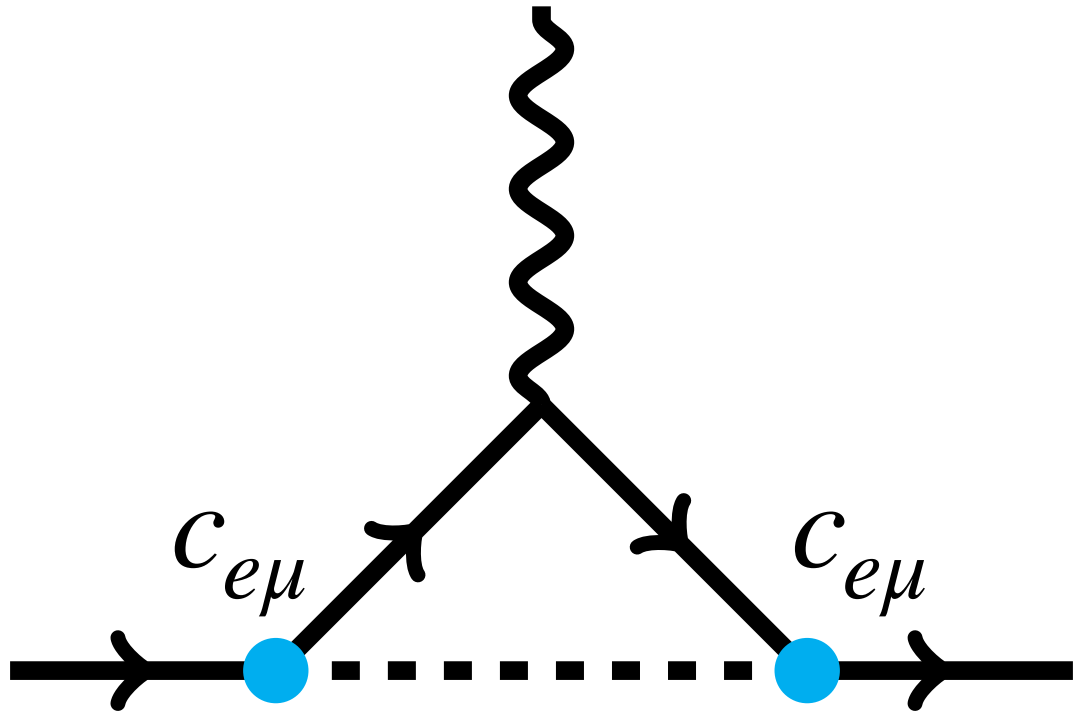
$$m_a \gtrsim 2\,\mathrm{GeV}, \quad c_{ee}/c_{\mu\mu} < 0,$$

$$\left|\frac{f_a}{c_{\mu\mu}}\right| \lesssim 100\,\mathrm{GeV}, \quad \left|\frac{f_a}{c_{ee}}\right| \lesssim 25\,\mathrm{GeV} \quad \text{for} \quad m_a = 5\,\mathrm{GeV}\ .$$

Explanation with off-diagonal couplings

Off-diagonal coupling, such as $c_{e\mu}$, was used to explain muon $g-2$, as well as electron $g-2$ based on measurement of α using Cs atoms (Parker et.al 2018).

Bauer, Neubert, Renner, Schnubel and Thamm 2019; Cornelia, Paradisi, Sumensari 2019.



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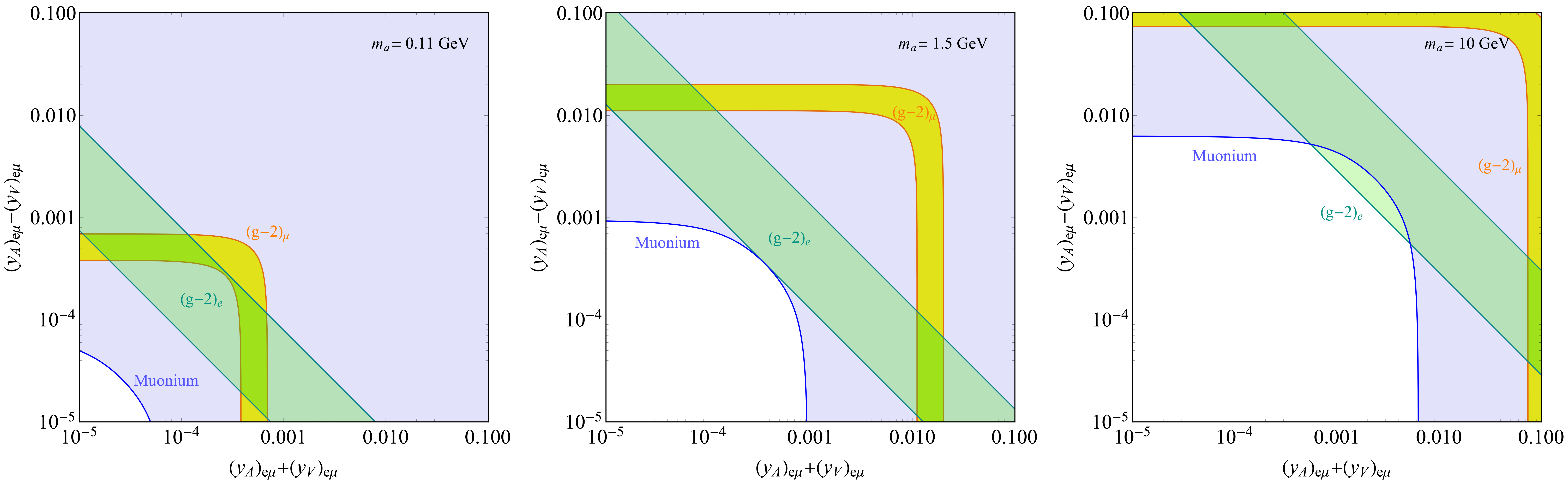
When $m_a > m_\mu$, $\Delta a_\mu \propto \dfrac{c_{e\mu}^2}{16\pi^2} > 0$

Electron $g-2$ anomaly is in doubt given the latest measurement of α using Rb atoms (Morel et.al 2020).

This explanation for muon $g-2$ is **ruled out** by muonium oscillation Endo, Iguro and Kitahara 2020 .



Recap

To use flavor-diagonal axion-coupling to the SM particles (i.e., muon and electron/photon) to explain muon $g - 2$, we need them to be huge:

$$\frac{f_a}{|c|} \lesssim 25\,\mathrm{GeV}$$

Cutoff of the axion EFT $4\pi f_a$ is also low.

Enhancing axion-photon couplings

$f_a/|c_{\gamma\gamma}| \lesssim (10-25)\,\mathrm{GeV} \Rightarrow$ new d.o.f, such as charged matter, with masses of $\mathcal{O}(10-25)\,\mathrm{GeV}$. E.g. KSVZ model where one integrates out heavy charged *vector-like fermions* with masses of order f_a , assuming that these fermions have order one charges and PQ charges. Kim, Shifman, Vainshtein, Zakharov 1980

1. <u>Large charges</u>

    — Landau pole of $U(1)_Y$: hypercharge $\lesssim 6$ for pole above Planck scale;

    — Highly charged fermions may not decay quickly.

2. Large PQ charges

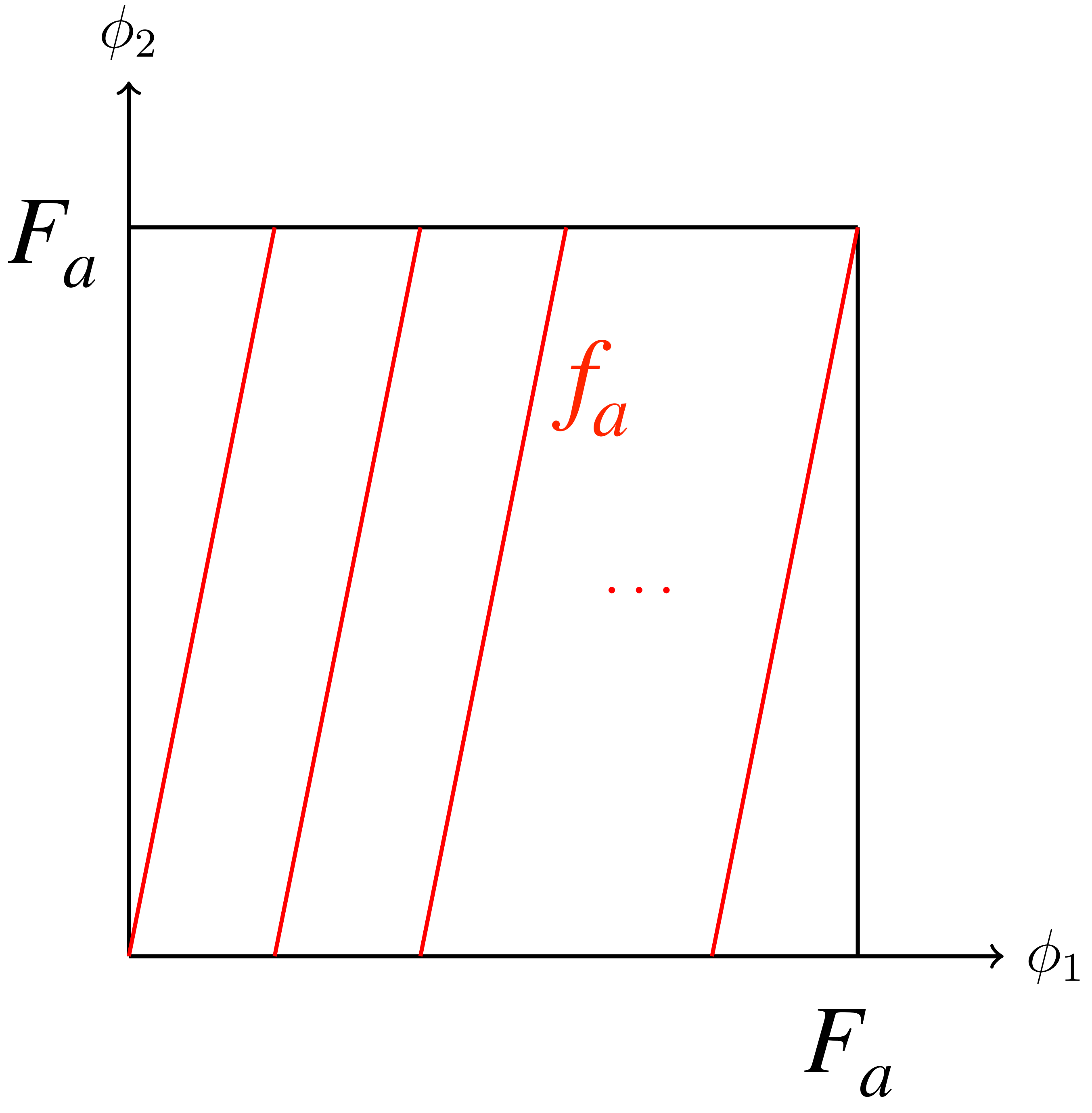
Masses of the fermions are **exponentially** suppressed

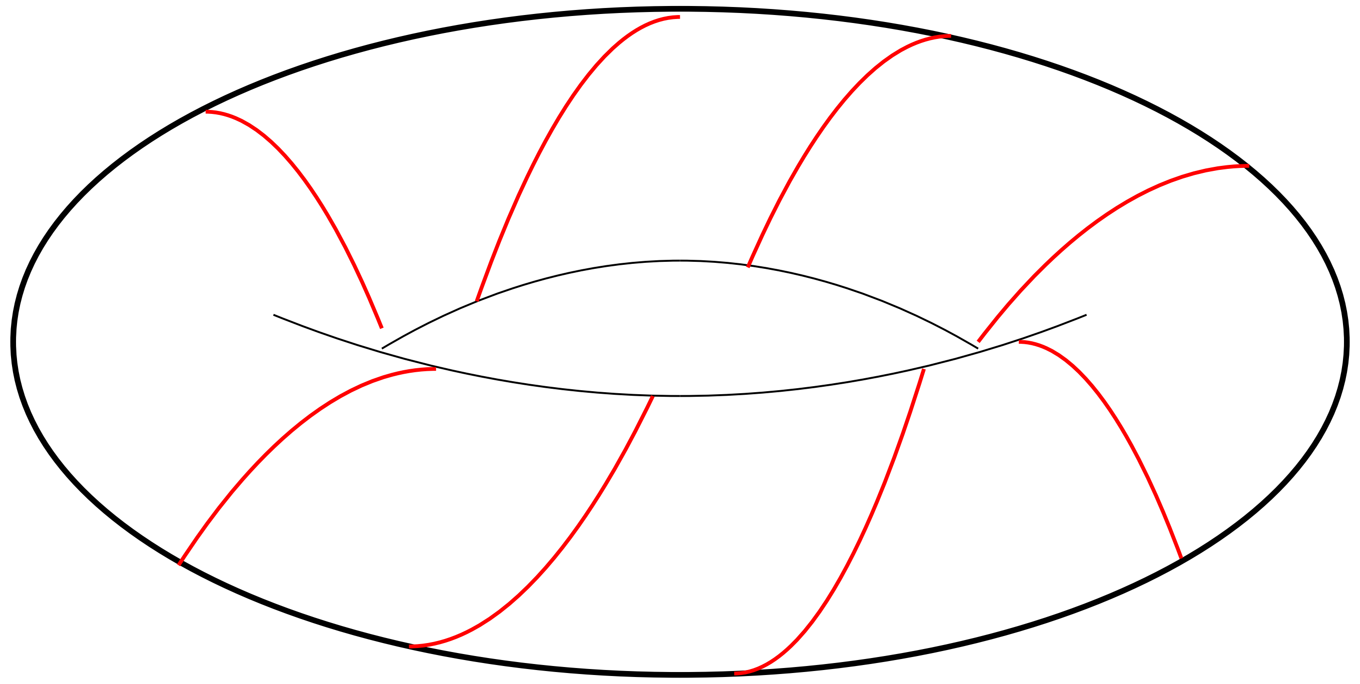
E.g.: $\dfrac{\phi^m}{\Lambda^{m-1}} L\tilde{L}$,

ϕ: PQ scalar with VEV f_a and unit PQ charge, $L(\tilde{L})$: vector-like leptons , m: PQ charge of $L(\tilde{L})$. Mass of $L(\tilde{L})$: $f_a \left(\dfrac{f_a}{\Lambda}\right)^{m-1}$.

One could UV complete it by adding a chain of vector-like fermions or clockworking. Agrawal, Fan, Reece and Wang 2017

3. <u>Alignment/clockwork mechanism:</u> enhance f_a of the axion compared to the fundamental period F_a . Dvali 2007; Kim, Nilles, Peloso 2014; Choi, Im; Kaplan, Rattazzi 2015





Choi, Kim, Yun '14

Applications to get photophilic axion: Farina, Pappadopulo, Rompineve, Tesi 2016; Agrawal, Fan, Reece, Wang 2017.

In theses models (mostly KSVZ-type), the fundamental period of the axion

$$F_a \sim f_a / \left| c_{\gamma\gamma} \right| \sim \mathcal{O}(10 - 25)\,\mathrm{GeV}$$

while f_a could be much larger. But one still needs new charged matter with masses of order F_a in these models.

A possible exception: clockworking n Higgses ($n \sim \mathcal{O}(10)$) in the DFSZ-type model (Dine, Fischler, Srednicki, Zhitnitsky 1981) so that the Higgs coupling to one flavor of SM fermion has a huge PQ charge $\sim 2^n$. Darme et.al 2020

Enhancing axion-fermion couplings

KSVZ-type model: mixing with vector-like fermions

SM right-handed lepton

$$\mathscr{L} \supset \chi^\dagger(i\bar{\sigma}^\mu\partial_\mu)\chi - y\,\tilde{\chi}\,\Phi_s\,E^c - M\,\tilde{\chi}\chi$$

$\chi, \tilde{\chi}$: heavy vector-like fermions with mass M

PQ scalar: $\Phi_s = f_a/\sqrt{2}\,e^{ia/f_a}$

Integrating out χ: $\chi = -\dfrac{y\Phi_s E^c}{M}$, we have $-\left|\dfrac{y f_a}{M}\right|^2 \dfrac{\partial_\mu a}{2f_a} E^{c\dagger}\bar{\sigma}^\mu E^c$. Then to explain muon g-2, $M \lesssim 125\,\mathrm{GeV}\left(\dfrac{y}{\sqrt{4\pi}}\right)\left(\dfrac{1/25\,\mathrm{GeV}}{c_{\mu\mu}/f_a}\right)$.

Three issues:

1. Always generate axion-fermion couplings of the **same** sign (case 2 doesn't work; case 1 might work but need to introduce new matter to generate large axion-photon couplings).

2. Light vector-like fermions with mass $\sim \mathcal{O}(100)$ GeV;

3. The vector-like fermions have sizable contributions to muon $g-2$.

$$\Delta a_\mu \approx 11 \times 10^{-10} y^2 \left(\frac{100\,\mathrm{GeV}}{M} \right)^2$$

DFSZ model: two Higgs doublets with PQ charges

Field	$SU(2)_L$	$U(1)_Y$	$U(1)_{\mathrm{PQ}}$	VEV
H_l	$\mathbf{2}$	$-\frac{1}{2}$	1	v_l
H_q	$\mathbf{2}$	$\frac{1}{2}$	1	v_q
Φ	$\mathbf{1}$	0	1	v_Φ

$$V_0(|H_l|, |H_q|, |\Phi|, |H_l H_q|) + \left(\lambda_{ql\Phi} H_l H_q \Phi^{\dagger 2} + y_u H_q Q U^c + y_d H_q^\dagger Q D^c + y_e H_l L E^c + \text{h.c.}\right).$$

$$\mapsto \frac{2}{f_a}\frac{v_q^2}{v_{\mathrm{EW}}^2}(\partial_\mu a) E^{c\dagger}\bar{\sigma}^\mu E^c + \frac{2}{f_a}\frac{v_l^2}{v_{\mathrm{EW}}^2}(\partial_\mu a)\left(U^{c\dagger}\bar{\sigma}^\mu U^c - D^{c\dagger}\bar{\sigma}^\mu D^c\right).$$

large coupling to leptons: $f_a \sim v_\Phi \lesssim 25\,\mathrm{GeV},\ v_q \sim v_{\mathrm{EW}}$

suppressed coupling to quarks: $v_l \ll v_{\mathrm{EW}}$

coupling to photons: $\frac{6}{f_a}\frac{\alpha}{4\pi} a F_{\mu\nu}\tilde{F}^{\mu\nu}$

Three issues:

1. Always generate axion-fermion couplings and axion-photon coupling of the **same** sign;

2. Light charged Higgs boson and light radial mode of Φ mixes with the Higgs: $h \to aa$, $a \to \gamma\gamma, \ell^+\ell^-$;

3. Additional Higgs bosons contribute to muon g-2 as well.

A possible exception as before: clockworking n Higgses $(n \sim \mathcal{O}(10))$ so that the Higgs coupling to the lepton has a huge PQ charge 2^n. The additional Higgses are relatively heavy and have suppressed VEV

$$v_k \sim \frac{v_{k-1}^2}{m_k^2} v_{\mathrm{EW}}, \quad k = 2, \cdots n$$

Darme et.al 2020

Part 1 Conclusion

A heavy axion-like particle with couplings to leptons and photons provides a tantalizing potential solution to muon $g-2$.

Yet to get these couplings, usually new states (charged or neutral but mixed with the Higgs boson) with masses of order a few 10's to a few 100's of GeV have to be present.

It is not a no-go theorem, but suggests that to consider an axion's contribution to muon $g-2$, one needs to consider more complete models specifying the origins of the couplings and other relevant d.o.f. These new d.o.f. could be strongly constrained and also contribute to muon $g-2$.

Part II. Axion mass from monopole loops

arXiv:2105.09950 with Fan, Fraser, Stout

New origin of axion potential

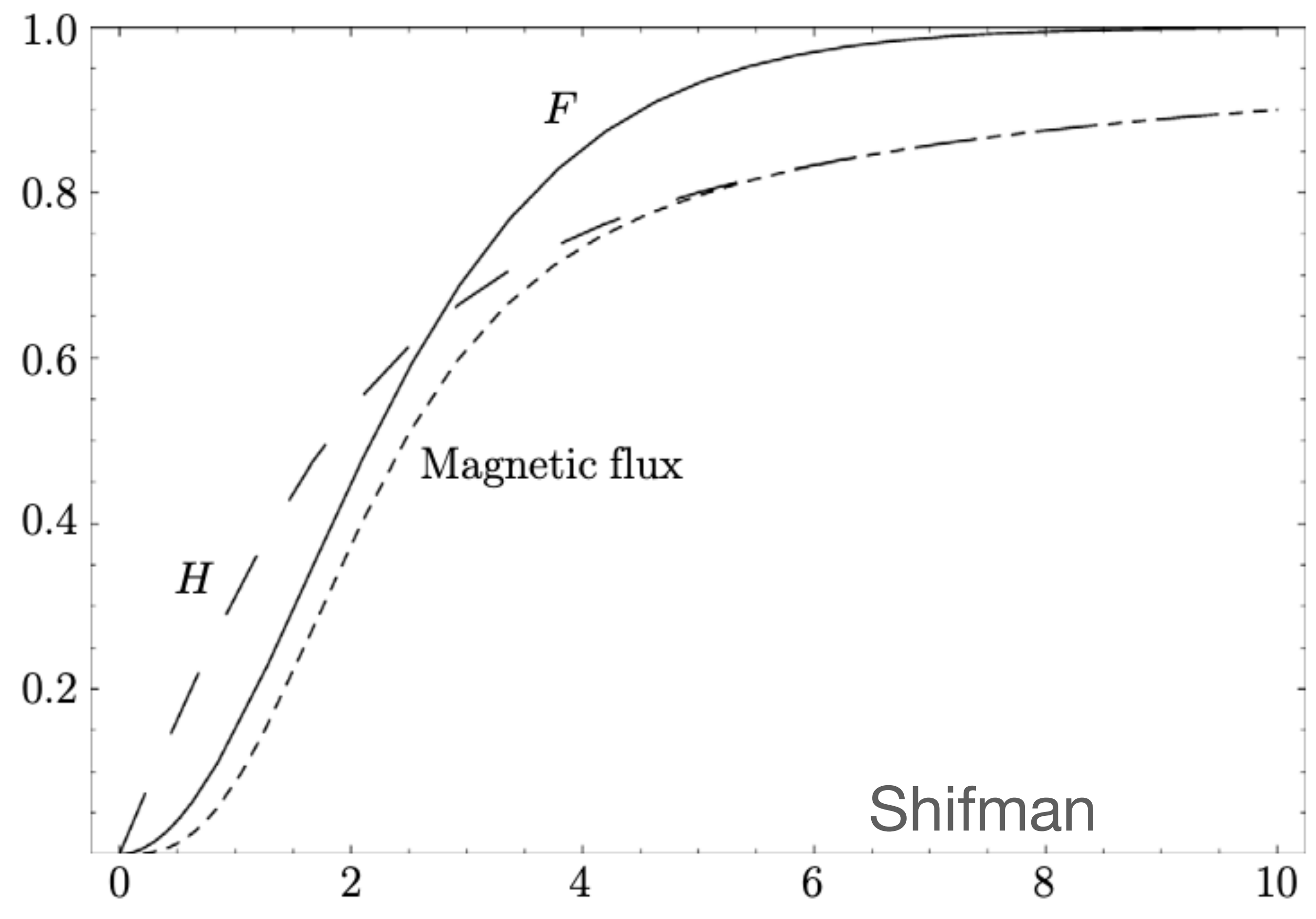
It is well known that for axion coupling to non-Abelian gauge group, instantons generate a potential for axion.

Yet for axion coupling to ***Abelian*** gauge fields, axion could still acquire a potential through ***loops of magnetic monopoles***.
Fan, Fraser, Reece and Stout 2021

Existence of magnetic monopoles: "***completeness hypothesis***"
Polchinski 2003

Monopole refresher: 't Hooft-Polyakov

$SU(2) \to U(1)$ symmetry broken by an adjoint vev: classical solution of 't Hooft-Polyakov ('t H-P) monopole.



$$\phi^a = v\hat{r}^a H(r), \quad A_i^a = \epsilon^{aij}\frac{1}{r}\hat{r}^j F(r)$$

$$r \to \infty : \quad H(r) \to 1,\, F(r) \to 1$$

$$r \to 0 : \quad H(r) \to 0,\, F(r) \to 0$$

The solution has **4 zero modes** (collective coordinates): 3 translations, 1 U(1) (large gauge transformation, not vanishing at infinity).

review: Shifman, *Advanced Topics in Quantum Field Theory*, Chapter 4

Possible charged states: not only magnetic monopoles, but also ***dyons*** (particles with both magnetic and electric charges).

E.g., in 't H-P case, a residual unbroken global $U(1)$ rotation could be realized by a compact real scalar. In 4d, this is described by QM of a particle living on a circle, $\sigma \cong \sigma + 2\pi$ (dyonic collective coordinate). This has a spectrum labelled by integers. The ground state is the magnetic monopole (with no electric charge) and the excited states are dyons.

Dyon tower

$\vdots$

$\pm 3e$

$\pm 2e$ excited states $m_n^2 = m_M^2 + m_\Delta^2 n^2$

$\pm e$

$0e$ ground state $m_0^2 = m_M^2$

Witten effect

Given $\frac{e^2\theta}{8\pi^2} F \wedge F = \frac{e^2\theta}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}$ ($\theta = a/f_a$ and e: unit of electric charge) and a point magnetic monopole (no electric charge when $\theta = 0$) at the origin, the Maxwell equations are modified:

Magnetic Gauss' law: $\nabla \cdot \mathbf{B} = \frac{g_m}{4\pi}\delta(\mathbf{r})$, g_m: unit of magnetic charge; $eg_m = 2\pi$ due to Dirac quantization condition;

Electric Gauss' law: $\nabla \cdot \mathbf{E} + \frac{e^2}{4\pi^2}\theta(\nabla \cdot \mathbf{B}) = 0 \;\Rightarrow\; \boxed{\frac{Q_E}{e} = -\frac{\theta}{2\pi}}$

A monopole obtains an effective electric charge in the presence of an axion background!

Witten, 1979

In general, the dyon electric charge is shifted to be

$$\frac{Q_E}{e} = n - \frac{\theta}{2\pi}, \quad n = 0, \pm 1, \pm 2, \cdots$$

The corresponding energy spectrum will be modified as well!

$$L = \frac{1}{2}\dot{\sigma}^2 + \frac{\theta}{2\pi}\dot{\sigma}$$

σ: dyonic collective coordinate

Conjugate momentum: $\Pi_\sigma = \dot{\sigma} + \dfrac{\theta}{2\pi}$

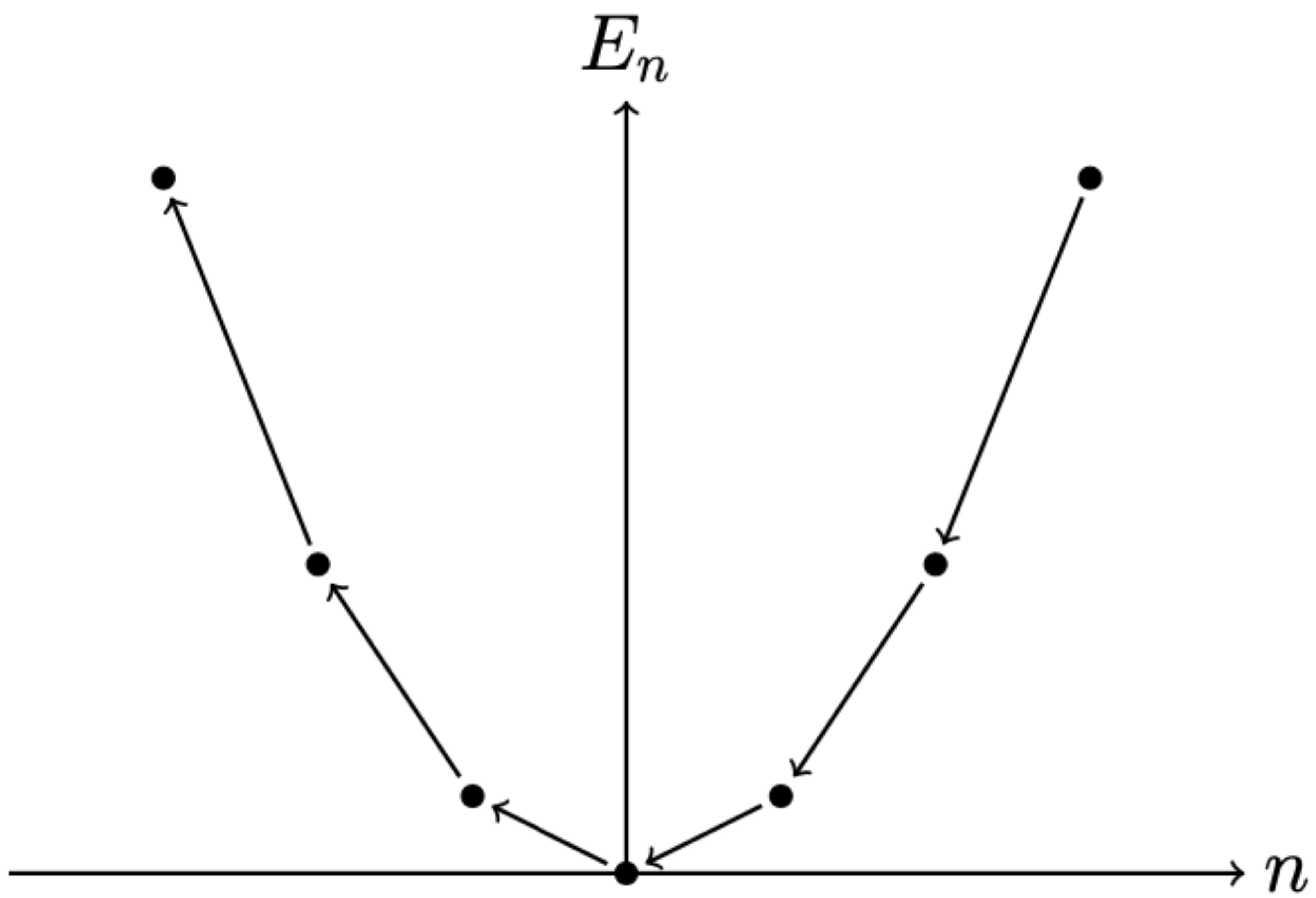
Hamiltonian: $H = \dfrac{1}{2}\left(\Pi_\sigma - \dfrac{\theta}{2\pi}\right)^2 \quad \Rightarrow \quad E_n = \dfrac{1}{2}\left(n - \dfrac{\theta}{2\pi}\right)^2$

$$\frac{1}{2}\left(-i\partial_\sigma - \frac{\theta}{2\pi}\right)^2 \psi_n = E_n \psi_n$$

The corresponding energy spectrum

$$m_n^2 - m_M^2 = m_\Delta^2 \boxed{\left(n - \frac{\theta}{2\pi}\right)^2}$$

ground state monopole mass at $\theta = 0$



periodicity through "monodromy" or rearrangement of the eigenstates:

$$n \to n+1, \quad \theta \to \theta + 2\pi$$

Integrating out these states $\Rightarrow$ vacuum potential for the axion θ!

Side note: ***different*** from the axion potential generated by ***monopole and anti-monopole plasma***! Fischler, Preskill 1983; Kawasaki et.al; Nomura et.al. 2015…

A plasma of monopoles and anti-monopoles could be generated through Kibble-Zurek mechanism in the early Universe.

Here we talk about the axion potential from the ***virtual*** effects of monopole (dyon) loops.

Our calculation can be carried out from two viewpoints:

1. Integrate out the dyons to get a Coleman-Weinberg potential for axion.
2. Do the path integral over all monopole loops.

Related by Poisson resummation

invariant length

transition amplitude $\langle x|x\rangle_\tau$

$$V_{\rm eff} = -\int_0^\infty \frac{d\tau}{2\tau}\frac{1}{2(2\pi\tau)^2}\exp\left(-\frac{m^2\tau}{2}\right)$$

$$m_n^2 = m_{\rm M}^2 + m_\Delta^2\left(n - \frac{\theta}{2\pi}\right)^2$$

$$-\sum_{n\in\mathbb{Z}}\int_0^\infty \frac{d\tau}{4\tau\,(2\pi\tau)^2}\exp\left(-\frac{m_{\rm M}^2\tau}{2} - \frac{m_\Delta^2\tau}{2}\left(n-\frac{\theta}{2\pi}\right)^2\right).$$

Poisson resum

$$\sum_{n\in\mathbb{Z}} e^{-\frac{1}{2}m_\Delta^2\tau\left(n-\frac{\theta}{2\pi}\right)^2} = \sum_{\ell\in\mathbb{Z}}\sqrt{\frac{2\pi}{m_\Delta^2\tau}}\exp\left(-\frac{2\pi^2\ell^2}{m_\Delta^2\tau} + i\ell\theta\right).$$

$$V_{\text{eff}}(\theta) = -\sum_{\boxed{\ell}=1}^{\infty} \frac{m_\Delta^2 m_{\mathrm{M}}^2}{32\pi^4 \ell^3} \mathrm{e}^{-2\pi\ell m_{\mathrm{M}}/m_\Delta} \cos(\ell\theta) \times \left(1 + \frac{3m_\Delta}{2\pi\ell m_{\mathrm{M}}} + \frac{3m_\Delta^2}{(2\pi\ell m_{\mathrm{M}})^2}\right),$$

winding number in the σ direction

Fan, Fraser, Reece and Stout, 2021

$$V_{\text{eff}}(\theta) = -\sum_{\boxed{\ell}=1}^{\infty} \frac{m_\Delta^2 m_{\text{M}}^2}{32\pi^4 \ell^3} e^{-2\pi\ell m_{\text{M}}/m_\Delta} \cos(\ell\theta) \times \left(1 + \frac{3m_\Delta}{2\pi\ell m_{\text{M}}} + \frac{3m_\Delta^2}{(2\pi\ell m_{\text{M}})^2}\right),$$

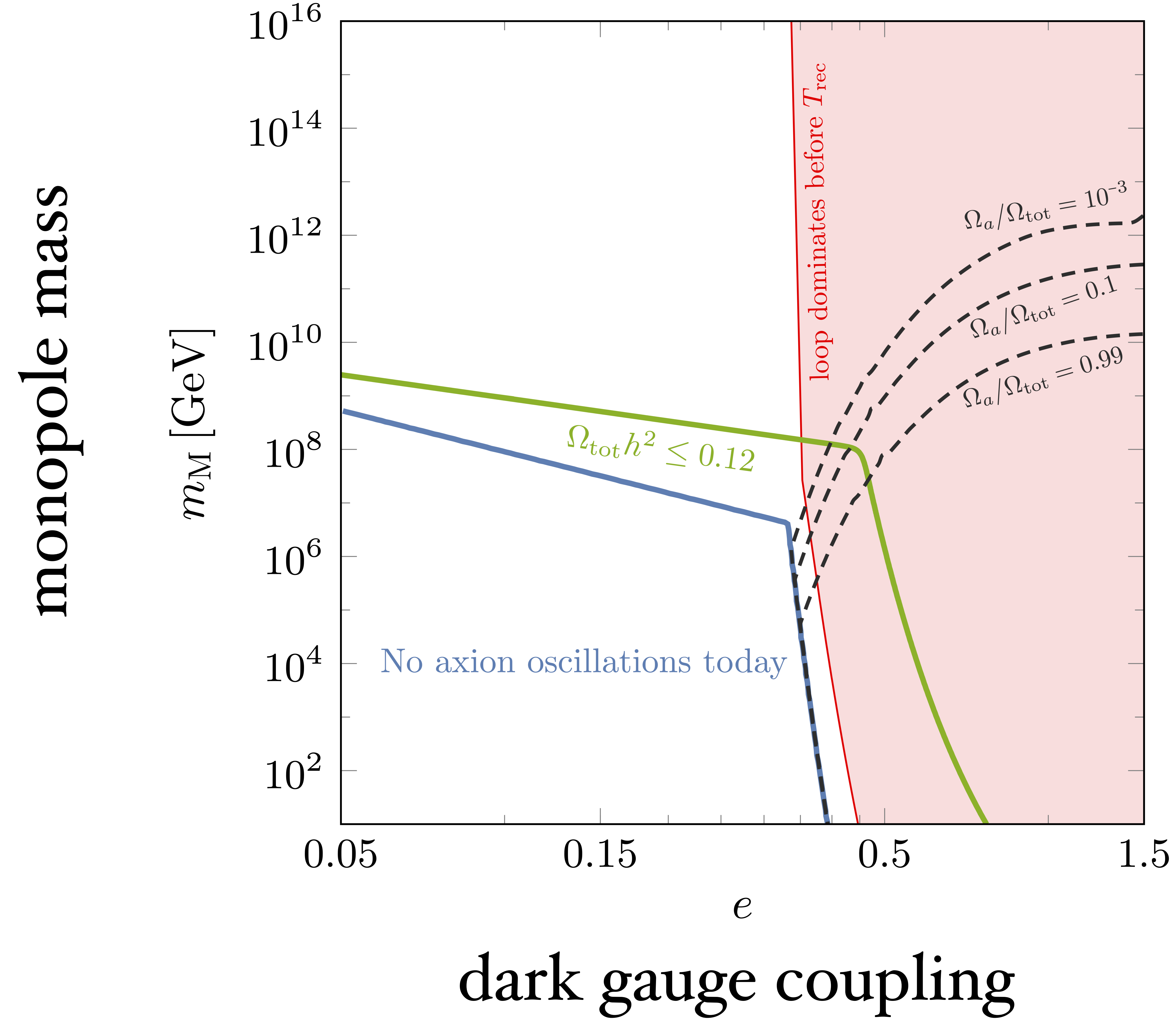
winding number in the σ direction

$e^{-S_{\text{inst}}} \sim e^{-8\pi^2/g^2}$ in 't H-P model: same instanton action as in YM theory!

Fan, Fraser, Reece and Stout, 2021

In a hidden gauged $U(1)$ sector with an axion and monopoles: both axion and monopole contribute to DM

$$m_a(T) = m_a^{\text{loop}} + m_a^{\text{plasma}}(T)$$



Conclusions

— Axion-like particles may appear in models of the muon $g-2$ anomaly, but we need a complete model beyond the axion EFT to have a full explanation.

— There is new source of axion potential from the axion coupling to Abelian gauge fields, via loops of magnetic monopoles. This should lead to a minimum mass to any axion coupling to photons (work in progress).

Still a lot to explore—both phenomenology and QFT—in axion physics!

BACKUP

Effective theory of axion and monopoles

$$S = \int \left[\frac{1}{2} f^2 \mathrm{d}\theta \wedge \star \mathrm{d}\theta - \frac{1}{2e^2} F \wedge \star F + \frac{k\theta}{8\pi^2} F \wedge F \right].$$

σ : dyonic collective coordinate, $\sigma \cong \sigma + 2\pi$

$$S = \int_\gamma \left[\frac{1}{2} l_\sigma \mathrm{d}_A\sigma \wedge \star \mathrm{d}_A\sigma + \frac{\theta}{2\pi} \mathrm{d}_A\sigma \right]$$

$$\mathrm{d}_A\sigma \;\equiv\; \mathrm{d}\sigma \,+\, kA$$

$$E_n = \frac{1}{2l_\sigma}\left(n - \frac{\theta}{2\pi}\right)^2. \quad l_\sigma \sim \frac{4\pi}{e^2k^2} r_*, \quad r_* = \max(r_c, r_0),$$

classical radius of magnetic monopole: $r_c = \pi/(e^2 m_M)$

length scale of axion screening: $r_0 = ke/(8\pi^2 f)$

Monopole loops

Euclidean path integral $\mathcal{Z}(\theta) = \sum_{\text{worldlines}} \int \mathcal{D}(\text{fields})\, \mathrm{e}^{-S_{\mathrm{E}}[\text{fields,worldlines},\theta]}$,

Effective potential $V_{\text{eff}}(\theta) = -\lim_{\mathcal{V}\to\infty} \frac{1}{\mathcal{V}} \log \mathcal{Z}(\theta)$.

Assuming that single monopole loop dominates

$$\mathcal{Z}(\theta) = \sum_{n=0}^{\infty} \frac{1}{n!} (Z_{\mathrm{S}^1})^n = \exp\big(Z_{\mathrm{S}^1}(\theta)\big) .$$

Sum over all paths that are topologically a circle S^1

invariant length: τ

$$Z_{\mathrm{S}^1} = \int_0^{\infty} \frac{\mathrm{d}\tau}{2\tau} Z(\tau, \theta),$$

sum over transition amplitudes at fixed θ of all trajectories with invariant length τ

$$\langle x'|x\rangle_\tau = \frac{1}{2(2\pi\tau)^2}\exp\left(-\frac{1}{2\tau}(x-x')^2 - m^2\tau\right)$$

$$V_{\text{eff}} = -\int_0^\infty \frac{\mathrm{d}\tau}{2\tau}\frac{1}{2(2\pi\tau)^2}\exp\left(-\frac{m^2\tau}{2}\right).$$

$$\Big\downarrow \quad m_n^2 = m_{\mathrm{M}}^2 + m_\Delta^2\left(n - \frac{\theta}{2\pi}\right)^2$$

$$-\sum_{n\in\mathbb{Z}}\int_0^\infty \frac{\mathrm{d}\tau}{4\tau\,(2\pi\tau)^2}\exp\left(-\frac{m_{\mathrm{M}}^2\tau}{2} - \frac{m_\Delta^2\tau}{2}\left(n-\frac{\theta}{2\pi}\right)^2\right).$$

Poisson resummation $\displaystyle\sum_{n\in\mathbb{Z}} \mathrm{e}^{-\frac{1}{2}m_\Delta^2\tau\left(n-\frac{\theta}{2\pi}\right)^2} = \sum_{\ell\in\mathbb{Z}}\sqrt{\frac{2\pi}{m_\Delta^2\tau}}\exp\left(-\frac{2\pi^2\ell^2}{m_\Delta^2\tau} + i\ell\theta\right).$

$$-\frac{\pi^2}{m_\Delta}\sum_{\ell\in\mathbb{Z}}\int_0^\infty \frac{\mathrm{d}\tau\,\mathrm{e}^{i\ell\theta}}{(2\pi\tau)^{7/2}}\exp\left(-\frac{m_{\mathrm{M}}^2\tau}{2}-\frac{2\pi^2\ell^2}{m_\Delta^2\tau}\right)$$

$$\Big\Downarrow$$

$$V_{\mathrm{eff}}(\theta) = -\sum_{\ell=1}^{\infty}\frac{m_\Delta^2 m_{\mathrm{M}}^2}{32\pi^4\ell^3}\mathrm{e}^{-2\pi\ell m_{\mathrm{M}}/m_\Delta}\cos(\ell\theta)\times \left(1+\frac{3m_\Delta}{2\pi\ell m_{\mathrm{M}}}+\frac{3m_\Delta^2}{(2\pi\ell m_{\mathrm{M}})^2}\right),$$

$$S_{\mathrm{M}} = m_{\mathrm{M}} \int_{\gamma} \mathrm{d}\lambda \sqrt{\frac{\mathrm{d}x_{\mu}}{\mathrm{d}\lambda}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda} + \frac{l_{\sigma}}{m_{\mathrm{M}}}\left(\frac{\mathrm{d}_A\sigma}{\mathrm{d}\lambda}\right)^2} + \int_{\gamma} \frac{\theta}{2\pi}\mathrm{d}_A\sigma\,,$$

$$\langle x', \sigma' | x, \sigma \rangle_{\tau} = \frac{1}{2(2\pi\tau)^{5/2}} \exp\left(-\frac{1}{2\tau}(x'-x)^2 - \frac{l_{\sigma}}{2m_{\mathrm{M}}\tau}(\sigma'-\sigma)^2 - \frac{m_{\mathrm{M}}^2\tau}{2} + \frac{i\theta}{2\pi}(\sigma'-\sigma)\right).$$

In the presence of a light fermion with $m \ll m_{\mathrm{M}}$, dyon's electric charge dispersed to a length $\sim 1/m$.

$$m_n^2 = m_{\mathrm{M}}^2 + m_\Delta^2 \left(n - \frac{\theta}{2\pi}\right)^2$$

$$m_\Delta^2 \sim m_{\mathrm{M}} m$$

$$m \to 0, \quad m_\Delta^2 \to 0$$