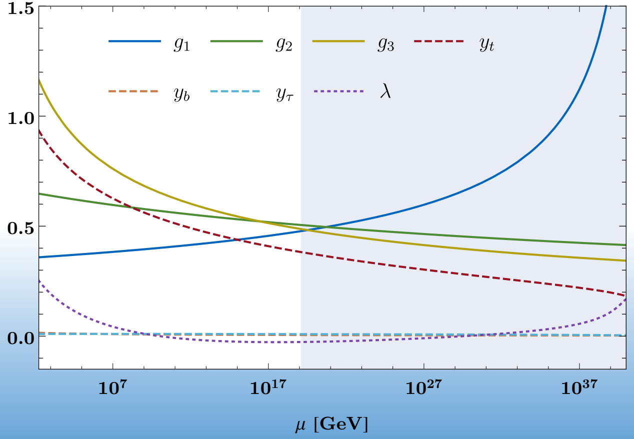
Divergences and Ambiguities in RG Functions



UNIVERSITÄT BERN

AEC
ALBERT EINSTEIN CENTER
FOR FUNDAMENTAL PHYSICS

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Outline

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Renormalization group flow

Callan–Symanzik equation for renormalized n-point functions:

$$\left(\frac{\partial}{\partial t} + \beta_g \frac{\partial}{\partial g} + n\gamma \right) G^{(n)}(\{p\}) = 0, \qquad \beta_I(g_I) = \frac{\mathrm{d}}{\mathrm{d}t} g_I \equiv \frac{\mathrm{d}}{\mathrm{d}\ln\mu} g_I$$

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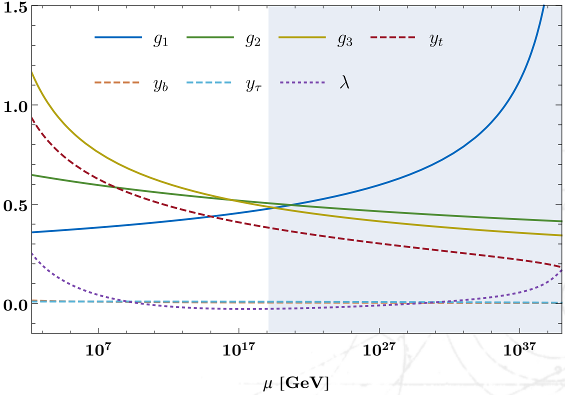
SM flow with 3rd generation Yukawa couplings:



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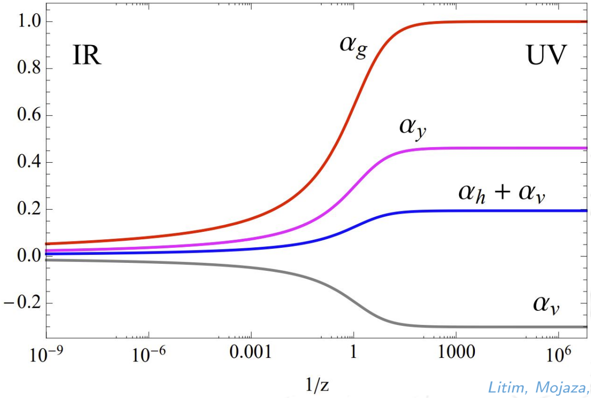
Asymptotic safety in the Litim-Sannino model:



Litim, Mojaza, Sannino [1501.03061]

Fixed points

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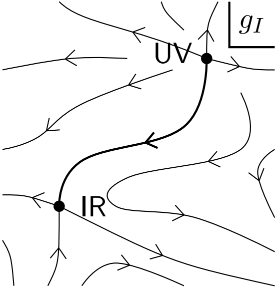

FPs are CFTs:

$$T^{\mu}{}_{\mu} = \beta_I \mathcal{O}^I = 0$$

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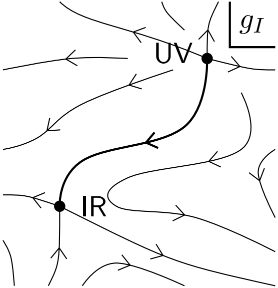

FPs are CFTs:

$$T^{\mu}{}_{\mu} = \beta_I \mathcal{O}^I = 0$$

The A-function of the Weyl anomaly imposes order on theory space:

- *The weak A-theorem*:

  Komargodski, Schwimmer [1107.3987]

  $$A_{\text{UV}} - A_{\text{IR}} \geq 0$$

- *The strong A-theorem*:

  $$\frac{\mathrm{d}}{\mathrm{d}\ln\mu} A \geq 0$$

- *The gradient flow*:

  Osborn '89
  Jack, Osborn '90

  $$\partial_I A \equiv \frac{\partial A}{\partial g^I} = T_{IJ}\beta^J$$

General β-functions in 4D QFTs

- General formulas for β-functions are known to order 3–2–2 ($\overline{\text{MS}}$).

  Macachek, Vaughn '83,'84, Pickering, Gracey, Jones [hep-ph/0104247]

- Computer packages with implementation of the general formulas: `SARAH 4`, `PyR@TE 3`, `ARGES`, `RGBeta`.

- Weyl Consistency Conditions from Osborn's Eq. establishes self-consistency constraints on β-functions. $\partial_I A = T_{IJ}\beta^J$

  Antipin *et al.* [1306.3234], Jack, Poole [1505.05400], Poole, AET [1906.04625]

- Work is being done on extracting the coefficients of the next loop order β-functions.

  Steudtner [2101.05823], Davies, Herren, AET [WIP]

γ-pole at the 3-loop order

Renormalization condition for 2-point functions: $(\overline{\text{MS}},\ d = 4 - \epsilon)$

$$Z^\dagger \text{—}\!\!\!\bigcirc\!\!\!\!\!\!\!\!\!\quad \text{1PI} \quad\text{—} Z \;+\; Z^\dagger \text{———} Z \;=\; \text{finite}, \qquad Z = \mathbb{1} + \sum_{n=1}^{\infty} \frac{z^{(n)}}{\epsilon^n}$$

with field anomalous dimension

$$\hat{\gamma} = Z^{-1} \frac{\mathrm{d}}{\mathrm{d}t} Z = \sum_{n=0}^{\infty} \frac{\hat{\gamma}^{(n)}}{\epsilon^n} \quad \Longrightarrow \quad \hat{\gamma}^{(0)} = -\zeta z^{(1)}, \qquad \zeta = k_I g_I \partial^I$$

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In SM $\hat{\gamma}^{(1)} \neq 0$ at the 3-loop order for $Z^\dagger = Z$

Bednyakov, Pikelner, Velizhanin [1406.7171] Herren, Mihaila, Steinhauser [1712.06614]

$$\hat{\gamma}^{(1)} - \hat{\gamma}^{(1)\dagger} = \left[z^{(1)},\, \zeta z^{(1)} \right]$$

$\hat{\gamma}^{(1)}$ can be made to vanish with $Z' = UZ$ for some divergent rotation U.

Outline

Four-dimensional QFT

Most general renormalizable theory in 4D (ignoring relevant couplings):

$$\begin{aligned}\mathcal{L} =& + \tfrac{1}{2}(D_\mu \phi)_a (D^\mu \phi)_a + i\psi_i^\dagger \bar{\sigma}^\mu (D_\mu \psi)^i + \mathcal{L}_{\mathrm{gh}} + \mathcal{L}_{\mathrm{gf}} \\ & - \tfrac{1}{4} a^{-1}_{AB} F^A_{\mu\nu} F^{B\mu\nu} - \tfrac{1}{2}\left(Y_{aij}\psi^i\psi^j + \mathrm{H.c.}\right)\phi_a - \tfrac{1}{24}\lambda_{abcd}\phi_a\phi_b\phi_c\phi_d\end{aligned}$$

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$$-\tfrac{1}{4}a^{-1}_{AB}F^A_{\mu\nu}F^{B\mu\nu} - \tfrac{1}{2}\left(Y_{aij}\psi^i\psi^j + \text{H.c.}\right)\phi_a - \tfrac{1}{24}\lambda_{abcd}\phi_a\phi_b\phi_c\phi_d$$

Compactly, the action is

$$S = S_{\text{kin}}[\Phi] + \int \mathrm{d}^d x \left(g_I\mathcal{O}^I(x) + \mathcal{J}_\alpha\Phi^\alpha\right)$$

The vacuum functional

$$e^{i\mathcal{W}[\mathcal{J}]} = \int [\mathcal{D}\Phi][\mathcal{D}A_\mu]e^{iS[\Phi,\,\mathcal{J}]}$$

generates all the connected n-point functions.

Local Renormalization Group

LRG was developed to probe the trace anomaly

$$[T^{\mu}{}_{\mu}] = \hat{\beta}_I[\mathcal{O}^I] + \hat{\upsilon} \cdot \partial_\mu [J_F^\mu]$$

with the path integral.

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Introduce new sources:

- $T_{\mu\nu}$: $\eta_{\mu\nu} \longrightarrow \gamma_{\mu\nu}(x)$
- $\mathcal{O}^I$: $g_I \longrightarrow g_I(x)$
- J_F^μ of the flavor symmetry G_F: introduce $a_\mu(x) \in \mathfrak{g}_F$

Jack, Osborn [1312.0428]
Baume, Keren-Zur, Rattazzi, Vitale [1401.5983]

To maintain local $\mathrm{Diff.} \times G \times G_F$ invariance

$$D_\mu = \partial_\mu - A_\mu \longrightarrow D_\mu = \nabla_\mu - A_\mu - a_\mu$$

Renormalization is easy

How can we renormalize the LRG action?

$$S_{\mathrm{kin}}[\Phi,\, \gamma,\, a] + \int \mathrm{d}^d x \sqrt{\gamma}\left(g_I \mathcal{O}^I + \mathcal{J}_\alpha \Phi^\alpha\right)$$

Composite renormalization: $[\phi^2] = Z_a \phi^2 + \mu^{-\epsilon} Z_b m^2 \phi + \mu^{-\epsilon} Z_c \partial^2 \phi$ (6D ϕ^3).

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$$S = S_{\mathrm{kin}}[\Phi,\, \gamma,\, a_0] + \int \mathrm{d}^d x \sqrt{\gamma}\left(g_{0,I} \mathcal{O}^I + \mathcal{J}_{0,\alpha} \Phi^\alpha\right) + S_{\mathrm{ct}}[\gamma,\, g_0,\, a_0]$$

e.g. $[\mathcal{O}^I] = \frac{\delta}{\delta g_I} S$

$$\mathcal{W}[\gamma,\, g,\, a,\, \mathcal{J}] = \mathcal{W}_0\big[\gamma,\, g_0(g),\, a_0(a, g),\, \mathcal{J}_0(\mathcal{J}, g)\big]$$

Weyl symmetry

S invariance under a local scaling: $\int \mathrm{d}^d x \sqrt{\gamma} \big(\frac{1}{2} (\partial_\mu \phi)^2 + \frac{d-2}{8(d-1)} \phi^2 R \big)$

$$\gamma_{\mu\nu} \longrightarrow e^{-2\sigma(x)} \gamma_{\mu\nu}, \qquad \Phi^\alpha \longrightarrow e^{\Delta_\alpha \sigma(x)} \Phi^\alpha$$

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Invariance of interactions in $d \neq 4$: $\int \mathrm{d}^d x \sqrt{\gamma}\, g_{0,I} \mathcal{O}^I$

$$g_{0,I} \longrightarrow g^\sigma_{0,I} = e^{k_I \epsilon \sigma} g_{0,I}$$

For the renormalized coupling, the coupling works as

$$g^\sigma_{0,I} = e^{\sigma k_I \epsilon} g_{0,I}(\mu, g(\mu)) = g_{0,I}(e^\sigma \mu, g(\mu)) = g_{0,I}(\mu, g(\mu e^{-\sigma}))$$

$$\Longrightarrow \delta_\sigma g_I = -\sigma \frac{\mathrm{d} g_I}{\mathrm{d}t} = -\sigma \hat{\beta}_I, \qquad t = \ln \mu$$

Weyl transformation

The infinitesimal generator of the Weyl transformation is

$$\Delta_\sigma^W = \int \mathrm{d}^d x \bigg(2\sigma \gamma^{\mu\nu} \frac{\delta}{\delta \gamma^{\mu\nu}} - \sigma \hat{\beta}_I \frac{\delta}{\delta g_I} + \sigma \mathcal{J}_\beta \big[(d - \Delta_\alpha) \delta^\beta{}_\alpha - \hat{\gamma}^\beta{}_\alpha \big] \frac{\delta}{\delta \mathcal{J}_\alpha} + \big[\partial_\mu \sigma \, \hat{\upsilon} - \sigma \, D_\mu g_I \, \hat{\rho}^I \big] \cdot \frac{\delta}{\delta a_\mu} \bigg)$$

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The symmetry is anomalous ($\Delta_\sigma^W S_{\mathrm{ct}} \neq 0$)

$$\Delta_\sigma^W \mathcal{W} = \int \mathrm{d}^d x\, \mathcal{A}_\sigma^W(\gamma,\, g,\, a)$$

The Weyl symmetry gives the trace anomaly equation

$$[T^\mu{}_\mu] = \hat{\beta}_I[\mathcal{O}^I] + \hat{\upsilon}\cdot\partial_\mu[J_F^\mu] - \eta_a\partial^2[\mathcal{O}_M^a] \qquad \text{(FSCC)}$$

RG transformation

Accounting identity for mass dimension:

$$\Delta^\mu \mathcal{W} = 0, \qquad \Delta^\mu = \mu \frac{\partial}{\partial \mu} + \int \mathrm{d}^d x \left(2\gamma^{\mu\nu} \frac{\delta}{\delta \gamma_{\mu\nu}} + (d - \Delta_\alpha) \mathcal{J}_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha} \right)$$

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The generator of the RG is $\Delta^{\mathrm{RG}} = \Delta^\mu - \Delta^W_{\sigma=1}$.

$$0 = \Delta^{\mathrm{RG}} \mathcal{W} = \left(\frac{\partial}{\partial t} + \hat{\beta}_I \partial^I + \int \mathrm{d}^d x \, \mathcal{J}_\beta \hat{\gamma}^\beta{}_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha} \right) \mathcal{W} \qquad \text{(FSCC)}$$

Exactly what we would get from $\mathrm{d}\mathcal{W} / \mathrm{d}t = 0$:

$$\left(\frac{\partial}{\partial t} + \beta_g \frac{\partial}{\partial g} + n\gamma \right) G^{(n)}(\{p\}) = 0$$

Flavor transformations

G_F is a symmetry of S,

$$\Delta^F_\omega = \int \mathrm{d}^d x \left(D_\mu \omega \cdot \frac{\delta}{\delta a_\mu} - (\omega\, g)_I \frac{\delta}{\delta g_I} - (\omega\, \mathcal{J})_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha} \right), \qquad \omega \in \mathfrak{g}_F,$$

but it is typically be anomalous: Keren-Zur [1406.0869]

$$\Delta^F_\omega \mathcal{W} = \int \mathrm{d}^d x\, \mathcal{A}^F_\omega(\gamma,\, g,\, a)$$

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$$\Delta^F_\omega \mathcal{W} = \int \mathrm{d}^d x\, \mathcal{A}^F_\omega(\gamma,\, g,\, a)$$

The Weyl generator can be combined with a flavor rotation to generate a class of Weyl symmetries:

$$\Delta^{W\,\prime}_\sigma = \Delta^W_\sigma + \Delta^F_{\sigma\alpha}, \qquad \alpha(g) \in \mathfrak{g}_F,$$

$$\left[\Delta^F_\omega,\, \Delta^{W\,\prime}_\sigma\right] = \left[\Delta^{W\,\prime}_\sigma,\, \Delta^{W\,\prime}_{\sigma'}\right] = 0, \qquad \Delta^{W\,\prime}_\sigma \mathcal{W} = \int \mathrm{d}^d x\, \mathcal{A}^{W\,\prime}_\sigma$$

A class of RG functions

Ambiguity in RG functions defined by the Weyl transformation:

$$\hat{\beta}'_I = \hat{\beta}_I + (\alpha\, g)_I, \quad \hat{\upsilon}' = \hat{\upsilon} + \alpha, \quad \rho'^I = \rho^I - \partial^I \alpha, \quad \hat{\gamma}'^{\alpha}{}_{\beta} = \hat{\gamma}^{\alpha}{}_{\beta} - \alpha^{\alpha}{}_{\beta}.$$

Flavor-improved RG functions are invariant:

$$B_I = \hat{\beta}_I - (\hat{\upsilon}\, g)_I, \qquad P^I = \hat{\rho}^I + \partial^I \hat{\upsilon}, \qquad \Gamma^{\alpha}{}_{\beta} = \hat{\gamma}^{\alpha}{}_{\beta} + \hat{\upsilon}^{\alpha}{}_{\beta}.$$

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We can choose a "gauge" where $\hat{\upsilon} = 0$:

$$\widehat{\Delta}^W_\sigma = \Delta^W_\sigma + \Delta^F{}_{-\sigma\,\hat{\upsilon}} = \int \mathrm{d}^d x \bigg(2\sigma \gamma^{\mu\nu} \frac{\delta}{\delta \gamma_{\mu\nu}} - \sigma B_I \frac{\delta}{\delta g_I} + \sigma \mathcal{J}_\beta \big[(d - \Delta_\alpha) \delta^\beta{}_\alpha - \Gamma^\beta{}_\alpha \big] \frac{\delta}{\delta \mathcal{J}_\alpha} - \sigma\, D_\mu g_I\, P^I \cdot \frac{\delta}{\delta a_\mu} \bigg)$$

B_I vs β_I

Limit Cycle

$$\hat{\beta}_I = (\hat{\upsilon}\, g)_I$$
$$B_I = 0$$

Limit cycles are actually CFTs

Fortin, Grinstein, Stergiou [1206.2921, 1208.3674]

$$\begin{aligned}[T^\mu{}_\mu] &= \hat{\beta}_I[\mathcal{O}^I] + \hat{\upsilon}\cdot\partial_\mu[J_F^\mu] \qquad \text{(FSCC)}\\ &= B_I[\mathcal{O}^I]\end{aligned}$$

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$$[T^{\mu}{}_{\mu}] = \hat{\beta}_I[\mathcal{O}^I] + \hat{\upsilon} \cdot \partial_\mu [J_F^\mu] \qquad \text{(FSCC)}$$
$$= B_I[\mathcal{O}^I]$$

B_I is a more physical β-function

Outline

How to compute RG functions

In $\overline{\text{MS}}$ the counterterms are arranged by poles

$$\delta g_I = \sum_{n=1}^{\infty} \frac{\delta g_I^{(n)}}{\epsilon^n}, \qquad Z = 1 + \sum_{n=1}^{\infty} \frac{z^{(n)}}{\epsilon^n}$$

The RG functions are determined recursively from the poles

$$\hat{\beta}_I^{(-1)} = -k_I g_I, \qquad \hat{\beta}_I^{(n)} = (\zeta - k_I)\delta g_I^{(n+1)} - \sum_{k=0}^{n-1} \hat{\beta}_J^{(k)} \partial^J \delta g_I^{(n-k)}, \qquad n \geq 0$$

$$\hat{\gamma}^{(n)} = -\zeta z^{(n+1)} + \sum_{k=0}^{n-1} \left[\hat{\beta}_I^{(k)} \partial^I z^{(n-k)} - z^{(n-k)} \hat{\gamma}^{(k)} \right], \qquad n \geq 0$$

Callan-Symanzik equation revisited

Evolution of *renormalized* amplitudes governed by CS Eq:

$$0 = \Delta^{\mathrm{RG}} \mathcal{W} = \left(\frac{\partial}{\partial t} + \hat{\beta}_I \partial^I + \int \mathrm{d}^d x \, \mathcal{J}_\beta \hat{\gamma}^\beta{}_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha} \right) \mathcal{W} \qquad \text{(FSCC)}$$

Callan-Symanzik equation revisited

Evolution of *renormalized* amplitudes governed by CS Eq:

$$\left(\frac{\partial}{\partial t} + \left(\epsilon \hat{\beta}_I^{(-1)} + \hat{\beta}_I^{(0)}\right)\partial^I + \int \mathrm{d}^d x\, \mathcal{J}_\beta \hat{\gamma}^{(0)\beta}{}_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha}\right)\mathcal{W}$$
$$= -\sum_{n=1}^{\infty} \frac{1}{\epsilon^n}\left(\hat{\beta}_I^{(n)}\partial^I + \int \mathrm{d}^d x\, \mathcal{J}_\beta \hat{\gamma}^{(n)\beta}{}_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha}\right)\mathcal{W}$$

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Recall our G_F Ward identity (FSCC):

$$0 = \Delta_\omega^F \mathcal{W} = \left((\omega\, g)_I \partial^I - \int \mathrm{d}^d x\, \mathcal{J}_\beta \omega^\beta{}_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha}\right)\mathcal{W}, \qquad \omega \in \mathfrak{g}_F$$

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RG finiteness (Conjecture)

$$\hat{\gamma}^{(n)} \in \mathfrak{g}_F \quad \text{and} \quad \hat{\beta}_I^{(n)} = -\left(\hat{\gamma}^{(n)}\, g\right)_I, \qquad n \geq 1$$

RG finiteness in the SM

At 3-loop RG divergences in the SM Herren, Mihaila, Steinhauser [1712.06614]

$$(4\pi)^6 \hat{\gamma}_q^{(1)} = \frac{g_1^2}{96}\big[y_u y_u^\dagger,\, y_d y_d^\dagger\big] + \frac{1}{32}\big[y_u y_u^\dagger y_u y_u^\dagger,\, y_d y_d^\dagger\big] + \frac{1}{32}\big[y_d y_d^\dagger y_d y_d^\dagger,\, y_u y_u^\dagger\big]$$

$$(4\pi)^6 \hat{\gamma}_u^{(1)} = \frac{1}{16} y_u^\dagger \big[y_d y_d^\dagger,\, y_u y_u^\dagger\big] y_u$$

$$(4\pi)^6 \hat{\beta}_{y_u}^{(1)} = -\frac{g_1^2}{96}\big[y_u y_u^\dagger,\, y_d y_d^\dagger\big] y_u - \frac{1}{32}\big[y_u y_u^\dagger y_u y_u^\dagger,\, y_d y_d^\dagger\big] y_u - \frac{1}{32}\big[y_d y_d^\dagger y_d y_d^\dagger,\, y_u y_u^\dagger\big] y_u + \frac{1}{16} y_u y_u^\dagger \big[y_d y_d^\dagger,\, y_u y_u^\dagger\big] y_u$$

$\hat{\beta}_{y_u}^{(1)} = -(\hat{\gamma}^{(1)}\, y_u)$, $\hat{\beta}_{y_u}^{(2)} = -(\hat{\gamma}^{(2)}\, y_u)$, *etc.* in the SM

$$(\omega\, y_u)^i{}_j = \omega_q{}^i{}_k y_u{}^k{}_j - y_u{}^i{}_k \omega_u{}^k{}_j + \omega_h y_u{}^i{}_j$$

Renormalization ambiguity

Consider a rotation with $R \in G_F$: $y_u \longrightarrow R_q y_u R_u^\dagger$

$$\mathcal{W}[\gamma,\, g,\, \mathcal{J},\, a] = \mathcal{W}[\gamma,\, Rg,\, R\mathcal{J},\, a^R] =$$
$$\mathcal{W}_0[\gamma,\, g_0,\, \mathcal{J}_0,\, a_0] = \mathcal{W}_0[\gamma,\, Rg_0,\, R\mathcal{J}_0,\, a_0^R], \quad (Rg_0)_I = g_{0,I}(Rg)$$

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$$\mathcal{W}_0[\gamma,\, g_0,\, \mathcal{J}_0,\, a_0] = \mathcal{W}_0[\gamma,\, Rg_0,\, R\mathcal{J}_0,\, a_0^R], \quad (Rg_0)_I = g_{0,I}(Rg)$$

Take a divergent rotation instead:

$$U = \exp\left[-\sum_{n=1}^{\infty} \frac{1}{\epsilon^n} u^{(n)}(g)\right], \qquad u^{(n)} \in \mathfrak{g}_F$$

$$\mathcal{W}[\gamma,\, g,\, \mathcal{J},\, a] = \mathcal{W}_0[\gamma,\, g_0,\, \mathcal{J}_0,\, a_0] = \mathcal{W}_0[\gamma,\, Ug_0,\, U\mathcal{J}_0,\, a_0^U]$$

Results in a change of Z: Ambiguity in taking $\sqrt{Z^\dagger Z}$

$$(U\mathcal{J}_0)_\alpha = \mathcal{J}_{0,\beta} U^{\dagger\beta}{}_\alpha = \mathcal{J}_\beta (Z^{-1}U^\dagger)^\beta{}_\alpha \implies \tilde{Z}^\alpha{}_\beta = U^\alpha{}_\gamma Z^\gamma{}_\beta.$$

Ambiguity in RG the functions

$\mathcal{W}_0[\gamma,\, g_0,\, \mathcal{J}_0,\, a_0] = \mathcal{W}_0[\gamma,\, U g_0,\, U\mathcal{J}_0,\, a_0^U]$ *but produce different RG functions!*

$$\Delta\gamma \equiv \hat{\gamma}^U - \hat{\gamma} = -\hat{\beta}_I U \partial^I U^\dagger$$
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  - $(\hat{\gamma}, \hat{\beta}_I)$ that are not RG finite cannot be made so
  - RG-finite $(\hat{\gamma}, \hat{\beta}_I)$ can be chosen simultaneously finite
- The flavor-improved RG functions are invariant:

$$B_I = \hat{\beta}_I - (\hat{\upsilon}\, g)_I, \qquad \Gamma = \hat{\gamma} + \hat{\upsilon}$$

Preferred set of RG functions

There is an infinite class of RG functions, which describe the flow:

$$\big(\hat{\gamma},\, \hat{\beta}_I,\, \hat{\upsilon}\big) \longrightarrow \big(\hat{\gamma} + \Delta\gamma,\, \hat{\beta}_I - (\Delta\gamma\, g),\, \hat{\upsilon} - \Delta\gamma\big), \qquad \Delta\gamma(g) \in \mathfrak{g}_F$$

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One additional consideration is

$$[T^{\mu}{}_{\mu}] = B_I[\mathcal{O}^I] - \eta_a \partial^2[\mathcal{O}^a_M] = \hat{\beta}_I[\mathcal{O}^I] + \hat{\upsilon}\cdot\partial_\mu[J^\mu_F] - \eta_a\partial^2[\mathcal{O}^a_M]$$

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Baume, Keren-Zur, Rattazzi, Vitale [1312.0428]

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Flavor-improved RG functions

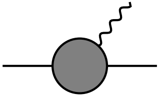
The set $\big(\Gamma,\, B_I,\, 0\big)$ is finite and unambiguous

Computing $\hat{\upsilon}$ in the SM

To determine $(\Gamma,\, B_I)$ one needs $\hat{\upsilon}$:

$$\hat{\upsilon} = N^I B_I, \qquad a_{0,\mu} = a_\mu + N^I D_\mu g_I$$

The N^I counterterm appears in e.g.

$$\left\langle Q_{\mathrm{L}} [J_F^\mu] \overline{Q}_{\mathrm{L}} \right\rangle_{1\mathrm{PI}} =$$ 

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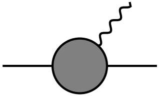
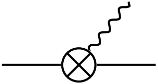
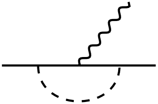
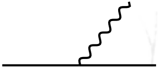


Sample 1-loop computation in the SM:

 $+$  $+\, Z^\dagger$  $Z = \text{finite}$

$$N_q^{(1)I} (t^A\, g)_I = \frac{1}{32\pi^2} \Big(y_u y_u^\dagger t_q^A - 2 y_u t_u^A y_u^\dagger + t_q^A y_u y_u^\dagger \Big) \;+\; (u \to d)$$

$$N_q^{(1)I} \hat{g}_I = \frac{1}{32\pi^2} \Big(\hat{y}_u y_u^\dagger - y_u \hat{y}_u^\dagger \Big) \;+\; (u \to d), \qquad (t^A\, y_u) = t_q^A y_u - y_u t_u^A$$

Computing $\hat{\upsilon}$ in the SM

- No contribution to $\hat{\upsilon}$ at 1- and 2-loop, as $\hat{\upsilon}^{(0)} \in \mathfrak{g}_F \implies \hat{\upsilon}^{(0)} = 0$.
- At 3-loop, we fixed 1983 parameters of N^I. For $n \geq 0$, we have

$$\Gamma_\ell^{(n)} = \Gamma_e^{(n)} = 0, \qquad \Gamma_u^{(n)} = \Gamma_d^{(n)} = 0, \qquad \Gamma_q^{(n)} \propto a_1 \left[y_u y_u^\dagger,\, y_d y_d^\dagger \right]$$

incomplete

- We overlooked a *single* counterterm because $(t^A\, a_1) = 0$:

$$N_q^I \hat{g}_I \supset \frac{C}{(4\pi)^6} \hat{a}_1 \left[y_u y_u^\dagger,\, y_d y_d^\dagger \right]$$

This is a counterterm of $\left\langle Q_{\mathrm{L}} [\mathcal{O}^I] \overline{Q}_{\mathrm{L}} \right\rangle_{1\mathrm{PI}}$ (WIP).

Summary

i) The occurrence of a certain class of ϵ poles in the RG functions is consistent with the Callan-Symanzik equation and not a sign of the theory or renormalization scheme breaking down.

ii) There is an ambiguity in choosing renormalization constants due to the flavor symmetry.

iii) Using the ambiguity, it is always possible to remove all the poles of the class discussed in i) simultaneously from $\hat{\gamma}$ and $\hat{\beta}_I$.

iv) The flavor-improved RG functions, (Γ, B_I), are unambiguous and finite and therefore a preferred choice.