Three-loop cusp anomalous dimension in QCD

A. Grozin, J. Henn, G. Korchemsky, P. Marquard

Why

IR behavior of scattering amplitudes $\rightarrow$ Wilson lines

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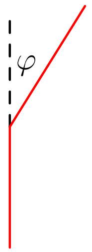
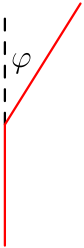
- $t\bar{t}$ production at LHC and ILC
- $t \rightarrow bW$
- $b \rightarrow c$
- $\dots$

Wilson lines



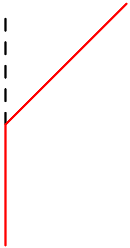
Wilson lines



Limiting cases

$\varphi \to 0$

$\varphi \to \infty$



$\varphi_E = \pi - \delta$

HQET heavy-to-heavy current

$$J = h_{v'}^{+} h_v = Z_J(\alpha_s(\mu); \varphi) J_r(\mu)$$
$$h_v = Z_h^{1/2}(\alpha_s(\mu)) h_{vr}(\mu)$$
$$\cosh \varphi = v \cdot v'$$

Green functions

$$-i<h_v(x)h_v^+(0)> = \delta(x_\perp)W(t) = Z_h\delta(x_\perp)W_r(t;\mu)$$

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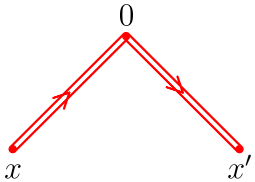


$$\begin{aligned}&(-i)^2 \langle h_{v'}(x') J(0) h_v^+(x)\rangle = \delta(x_\perp)\delta(x'_{\perp'}) W(t,t';\varphi)\\ &= Z_h Z_J \delta(x_\perp)\delta(x'_{\perp'}) W_r(t,t';\varphi;\mu)\end{aligned}$$

Renormalization

$$W(t,t';0) = W(t+t')$$

$$\log \frac{W(t,t';\varphi)}{W(t,t';0)} = \log Z_J + \text{finite}$$

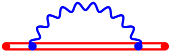
$$\Gamma(\alpha_s,\varphi) = \frac{d\log Z_J}{d\log\mu}$$

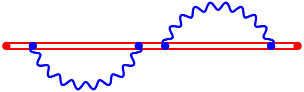
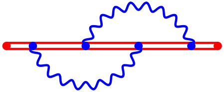
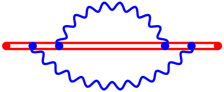
$$\Gamma(\alpha_s,0) = 0$$

Exponentiation in QED

Coordinate space, Wilson line of any shape ($n_f = 0$)

$W(t) =$ 

$+$ 

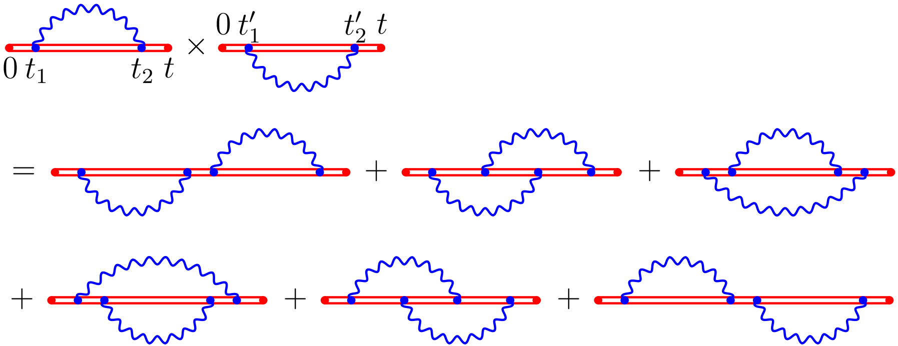
$+$  $+$  $+$ 

$+ \cdots$

$\log W(t) = \,?$

Exponentiation in QED

$$0 < t_1 < t_2 < t,\ 0 < t_1' < t_2' < t$$



Exponentiation in QED

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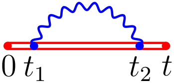
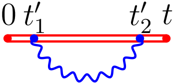
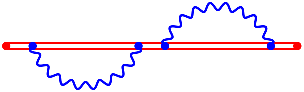
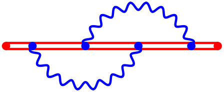
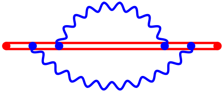
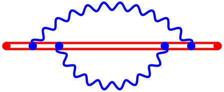
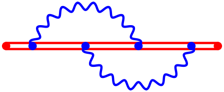
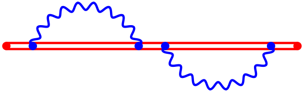
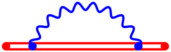


$\times$

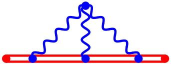


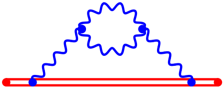
$=$  $+$  $+$ 

$+$  $+$  $+$ 

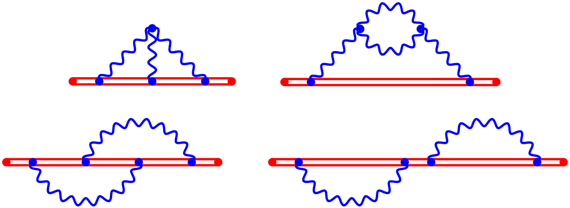
$\log W(t) =$ 

Exponentiation in QCD

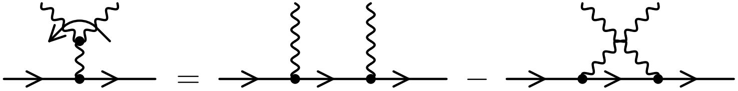




Exponentiation in QCD

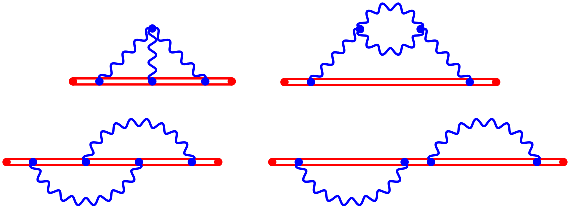


$[t^a, t^b] = i f^{abc} t^c$



Gatheral (1983); Frenkel, Taylor (1984)

Exponentiation in QCD



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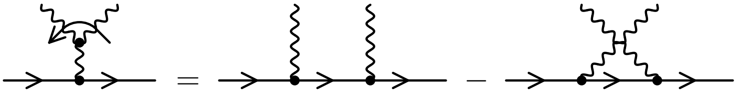

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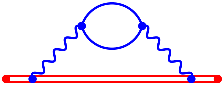

$T_F n_f \Rightarrow$ all color structures allowed

Exponentiation in QCD

$$\log W = C_F \frac{g_0^2}{(4\pi)^{d/2}} \Bigg[w + \left(C_A w_A + T_F n_f w_f\right) \frac{g_0^2}{(4\pi)^{d/2}}$$

$$+ \left(C_A^2 w_{AA} + C_F T_F n_f w_{Ff} + C_A T_F n_f w_{Af} + \left(T_F n_f\right)^2 w_{ff}\right) \left(\frac{g_0^2}{(4\pi)^{d/2}}\right)^2 \Bigg]$$

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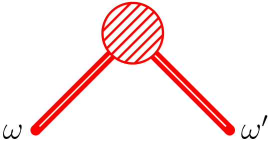
$$\Gamma = C_F \frac{\alpha_s}{\pi} \Bigg[\gamma + \left(C_A \gamma_A + T_F n_f \gamma_f\right) \frac{\alpha_s}{\pi}$$
$$+ \left(C_A^2 \gamma_{AA} + C_F T_F n_f \gamma_{Ff} + C_A T_F n_f \gamma_{Af} + \left(T_F n_f\right)^2 \gamma_{ff}\right) \left(\frac{\alpha_s}{\pi}\right)^2 \Bigg]$$

Momentum space



Vertex function V: 1PI, without external-leg propagators

$$G(\omega,\omega';\varphi) = V(\omega,\omega';\varphi)S_v(\omega)S_{v'}(\omega')$$
$$V(\omega,\omega';\varphi) = Z_J Z_h^{-1} V_r(\omega,\omega';\varphi;\mu)$$
$$\log V(\omega,\omega';\varphi) - \log V(\omega,\omega';0) = \log Z_J + \text{finite}$$

Convenient to set $\omega' = \omega$

$\varphi = 0$

$$V(\omega,\omega';0) = \frac{S^{-1}(\omega') - S^{-1}(\omega)}{\omega' - \omega} = Z_h^{-1} V_r(\omega,\omega';0;\mu)$$
$$\log V(\omega,\omega';0) = -\log Z_h + \text{finite}$$

Z_h is gauge dependent; Z_J is gauge invariant

History

1 loop

$$\Gamma(\alpha_s, \varphi) = C_F \frac{\alpha_s}{\pi} \left(\varphi \coth \varphi - 1\right)$$

Follows from the soft radiation function in classical electrodynamics

The Guinness Book of Records The anomalous dimension known for a longest time ($>$ 100 years)

2 loops Korchemsky, Radyushkin (1987)

3 loops here

History

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γ_h at 3 loops — Chetyrkin, Grozin (2003)

Abelian large n_f structures

$C_F(T_F n_f)^{L-1}\alpha_s^L$ and $C_F^2(T_F n_f)^{L-2}\alpha_s^L$ $(L \geq 3)$

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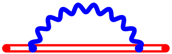
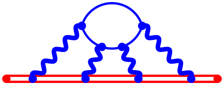
$C_F(T_F n_f)^{L-1}\alpha_s^L$ and $C_F^2(T_F n_f)^{L-2}\alpha_s^L$ $(L \geq 3)$

QED with n_f flavors $C_F = 1$, $C_A = 0$, $T_F = 1$, $\beta_0 = -\frac{4}{3}n_f$
At Lβ_0 and NLβ_0, the Wilson line of any shape

$\log W =$ 

with

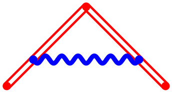
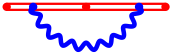
up to NLβ_0. First broken at NNLβ_0



$n_f^{L-3}\alpha^L$ $(L \geq 4)$

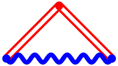
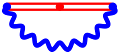
Abelian large n_f structures

$$\log W(t,t';\varphi) - \log W(t,t';0)$$

$=$  $-$  $= \log Z_J + \text{finite}$

(external-leg corrections cancel). Momentum space

$$\log V(\omega,\omega;\varphi) - \log V(\omega,\omega;0)$$

$=$  $-$  $= \log Z_J + \text{finite}$

Abelian large n_f structures

$L\beta_0$

$$\log\frac{V(\varphi)}{V(0)} = \frac{1}{\beta_0}\sum_{L=1}^{\infty} a_L b_0^L = \frac{1}{\beta_0}\sum_{L=1}^{\infty}\frac{F(\epsilon, L\epsilon)}{L}\left(\frac{b}{\epsilon+b}\right)^L$$

$$b_0 = \frac{\beta_0 e_0^2}{(4\pi)^{d/2}} = bZ_\alpha(b)e^{\gamma\epsilon}\mu^{2\epsilon} \qquad b = \frac{\beta_0\alpha}{4\pi} \qquad Z_\alpha = \frac{1}{1+b/\epsilon}$$

$$F(\epsilon, u) = \sum_{n,m=0}^{\infty} F_{nm}\epsilon^n u^m \qquad \log Z_J = \frac{Z_1}{\epsilon} + \frac{Z_2}{\epsilon^2} + \cdots$$

$$\beta_0 Z_1 = F_{00}b - F_{10}\frac{b^2}{2} + F_{20}\frac{b^3}{3} - F_{30}\frac{b^4}{4} + \cdots$$

$$\Gamma = -2\frac{dZ_1}{d\log b} = -2\frac{b}{\beta_0}F(-b, 0)$$

Abelian large n_f structures

$$\Gamma = C_F \frac{\alpha_s}{6\pi} \frac{\varphi \coth \varphi - 1}{B(2+b, 2+b)\Gamma(1+b)\Gamma(1-b)}$$
$$= C_F \frac{\alpha_s}{\pi} \left[1 + \frac{5}{3} b - \frac{1}{3} b^2 + \left(2\zeta_3 - \frac{1}{3} \right) b^3 + \cdots \right] (\varphi \coth \varphi - 1)$$

Beneke, Braun (1995)

Abelian large n_f structures

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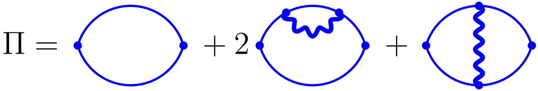
Beneke, Braun (1995)
NLβ_0

- NL a_L. Photon self energy



- NL Z_α

$C_F C_A T_F n_f$

$$\log Z_J = \cdots + C_F \left(\frac{\alpha_s}{\pi}\right)^3$$

$$\left[C_A^2 z_{AA} + T_F n_f (C_F z_{Ff} + C_A z_{Af}) + (T_F n_f)^2 z_{ff}\right]$$

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$$N_c \to \infty \qquad N_c \left(\frac{z_{Ff}}{2} + z_{Af}\right)$$

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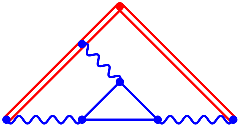
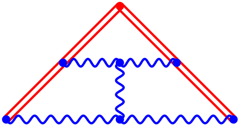


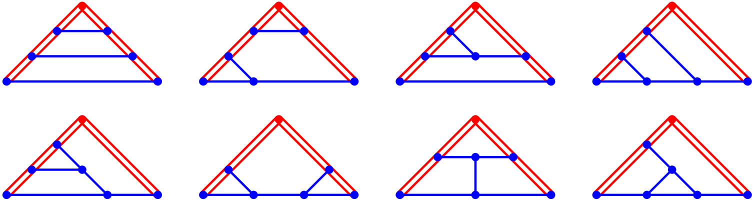
$C_F C_A^2$

$N_c \to \infty \qquad N_c z_{AA}$

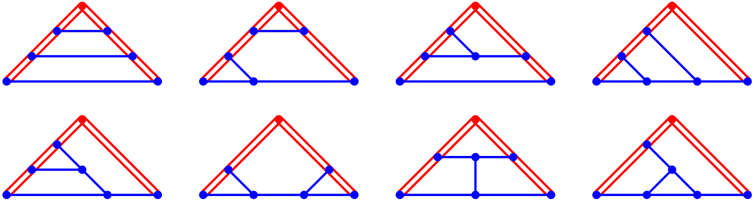


Only topologies surviving at $N_c \to \infty$

Topologies and master integrals



Topologies and master integrals



71 master integrals

- 7 straight-line [Grozin (2000)]
- 8 products of lower loops
- 10 generalized triangles [Grozin, Kotikov (2011)]
- 46 nontrivial

Differential equations

$$x = e^{-\varphi}$$

Symmetry $x \to 1/x$. Differentiate in x and reduce to masters.

Differential equations

$$x = e^{-\varphi}$$

Symmetry $x \to 1/x$. Differentiate in x and reduce to masters.
Canonical basis $\vec{f}$ [Henn (2013)]

$$\partial_x \vec{f}(x,\epsilon) = \epsilon \left[\frac{a}{x} + \frac{b}{x+1} + \frac{c}{x-1} \right] \vec{f}(x,\epsilon)$$

4 singular points

- $x = 1$ $(\varphi \to 0)$
- $x = 0,\ \infty$ $(\varphi \to \pm\infty)$
- $x = -1$ $(\varphi_E \to \pi)$

Harmonic polylogarithms $H_{n_1 \ldots n_k}(x)$ $(n_i = 0,\ \pm 1)$ [Remiddi,Vermaseren (2000)]
Uniform weight functions

Result

$$
\begin{aligned}
\Gamma(\alpha_s, x) = C_F \frac{\alpha_s}{\pi} \Bigg\{ & \tilde{A}_1 \\
& + \left[\frac{1}{2} C_A \left(\tilde{A}_3 + \tilde{A}_2 \right) + \frac{1}{9} \left(\frac{67}{4} C_A - 5 T_F n_f \right) \tilde{A}_1 \right] \frac{\alpha_s}{\pi} \\
& + \Bigg\{ \left[\frac{1}{4} \left(\tilde{A}_5 + \tilde{A}_4 + \tilde{B}_5 + \tilde{B}_3 \right) + \frac{67}{36} \tilde{A}_3 + \frac{29}{18} \tilde{A}_2 \right. \\
& \qquad \left. + \frac{1}{24} \left(11 \zeta_3 + \frac{245}{4} \right) \tilde{A}_1 \right] C_A^2 \\
& \quad - \left[\frac{5}{9} \left(\tilde{A}_3 + \tilde{A}_2 \right) + \frac{1}{6} \left(7 \zeta_3 + \frac{209}{36} \right) \tilde{A}_1 \right] C_A T_F n_f \\
& \quad + \left(\zeta_3 - \frac{55}{48} \right) \tilde{A}_1 C_F T_F n_f - \frac{1}{27} \tilde{A}_1 \left(T_F n_f \right)^2 \Bigg\} \left(\frac{\alpha_s}{\pi} \right)^2 \Bigg\}
\end{aligned}
$$

Result

$$\tilde{A}_i(x) = A_i(x) - A_i(1)$$

$$A_1(x) = \frac{\xi}{2} H_1(y)$$

$$A_2(x) = \frac{1}{2} H_{1,1}(y) + \frac{\pi^2}{3} - \xi \left[\frac{1}{2} H_{1,1}(y) - H_{1,0}(y) \right]$$

$$A_3(x) = -\xi \left[\frac{1}{4} H_{1,1,1}(y) + \frac{\pi^2}{6} H_1(y) \right] + \xi^2 \left[\frac{1}{4} H_{1,1,1}(y) + \frac{1}{2} H_{1,0,1}(y) \right]$$

$$\ldots$$

$$y = 1 - x^2 \qquad \xi - \frac{1 + x^2}{1 - x^2}$$

Uniform weight i

$\varphi \to \infty$

$x \to 0$

$$\Gamma(\alpha_s, x) = K(\alpha_s)\varphi + \mathcal{O}(1)$$

Korchemsky (1989); Korchemsky, Marchesini (1993)

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$x \to 0$

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Korchemsky (1989); Korchemsky, Marchesini (1993)

$$\begin{aligned}
K(\alpha_s) = C_F \frac{\alpha_s}{\pi} \Bigg\{ 1 &+ \left[\frac{1}{12} \left(\pi^2 - \frac{67}{3} \right) C_A + \frac{5}{9} T_F n_f \right] \frac{\alpha_s}{\pi} \\
&+ \left[\frac{1}{24} \left(\frac{11}{30}\pi^4 + 11\zeta_3 - \frac{67}{9}\pi^2 + \frac{245}{4} \right) C_A^2 \right. \\
&\quad - \frac{1}{6} \left(7\zeta_3 - \frac{5}{9}\pi^2 + \frac{209}{36} \right) C_A T_F n_f \\
&\quad \left. + \left(\zeta_3 - \frac{55}{48} \right) C_F T_F n_f - \frac{1}{27} (T_F n_f)^2 \right] \left(\frac{\alpha_s}{\pi} \right)^2 \Bigg\}
\end{aligned}$$

Moch, Vermaseren, Vogt (2004)

Let's define a

$$K(\alpha_s) = C_F \frac{a}{\pi}$$

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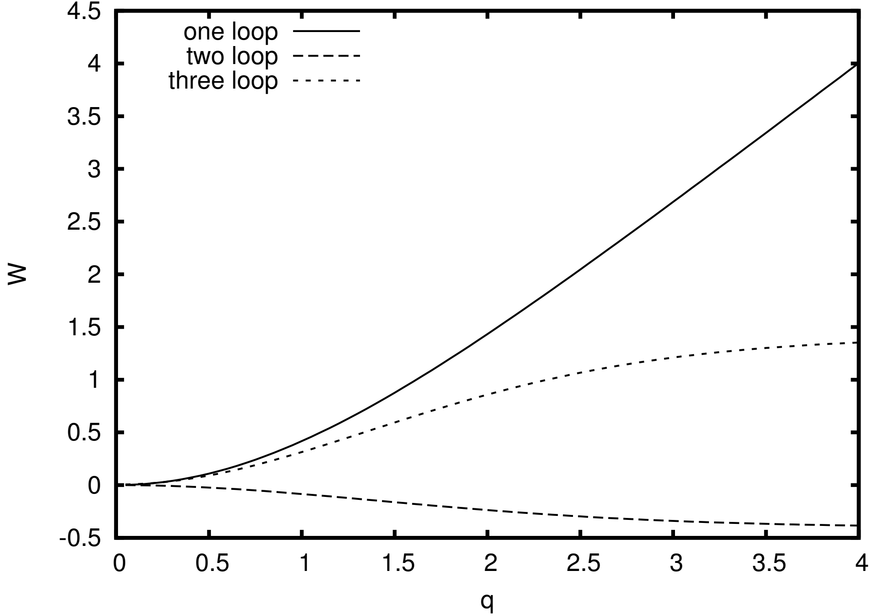
$$\Omega(a,x) = C_F \frac{a}{\pi}\left[\tilde{A}_1 + \frac{1}{2}\left(\tilde{A}_3 + \tilde{A}_2 + \frac{\pi^2}{6}\tilde{A}_1\right) C_A \frac{a}{\pi}\right.$$
$$\left. + \frac{1}{4}\left(\tilde{A}_5 + \tilde{A}_4 - \tilde{A}_2 + \tilde{B}_5 + \tilde{B}_3 + \frac{\pi^2}{3}\tilde{A}_3 + \frac{\pi^2}{3}\tilde{A}_2 - \frac{\pi^4}{180}\tilde{A}_1\right) C_A^2 \left(\frac{a}{\pi}\right)^2\right]$$

Does not contain n_f!



one loop
two loop
three loop
W
q
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
4

$\varphi_E \to \pi$

Euclidean $\varphi_E = \pi - \delta$

$$\Gamma = \frac{rV(r)}{\delta}$$

Kilian, Mannel, Ohl (1993)

Conformal symmetry

Euclidean space

$$ds^2 = dx_0^2 + d\vec{x}^2$$

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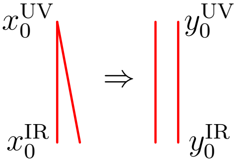

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Choosing the right μ in 1-loop Γ

At 3 loops $\delta\Gamma(\pi - \delta)$ differs from $rV(r)$ by a term $\sim \beta_0\alpha_s^3$

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- $\Omega(a, x)$ does not contain n_f: only 1 gluonic color structure at each L (1 structure at 2 loops and 3 structures at 3 loops disappear)
- $\varphi_E = \pi - \delta$: the relation to $V(r)$ which follows from conformal invariance is violated at 3 loops by a term $\sim \beta_0 \alpha_s^3$