Theory of the muon $g-2$

Why the 9th decimal place matters

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Outline

Part 1

Answer the questions:

- What is the anomalous magnetic moment $(g-2)$ of the muon ?
- Why does the 9th decimal place matter ?

Part 2

- $g = 2$ from Dirac equation, $g-2$ in Quantum Field Theory
- Electron $g-2$
  - Theory
  - Experiment at Harvard
  - Determination of the fine-structure constant α
- Muon $g-2$
  - Brookhaven Experiment
  - Theory: QED and Weak contributions
  - QCD / Hadronic contributions: vacuum polarization and light-by-light scattering
  - Test of the Standard Model: a sign of New Physics ?
- Conclusions and Outlook

Selected references

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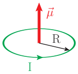
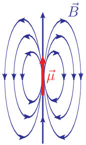
Part 1

Magnetic (dipole) moment



Magnetic dipole moment $\vec{\mu}$ of current loop:
$|\vec{\mu}| = I\, R^2 \pi =$ current $\times$ area



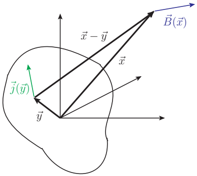
magnetic dipole field

For general, stationary current distribution $\vec{j}(\vec{y})$ $[\mathrm{A/m^2}]$ the magnetic dipole moment $\vec{\mu}$ determines the magnetic field $\vec{B}$ far away from the currents (from Biot-Savart's law):

$$\vec{B}(\vec{x}) = -\frac{\mu_0}{4\pi} \int d^3y \, \frac{\vec{x}-\vec{y}}{|\vec{x}-\vec{y}|^3} \times \vec{j}(\vec{y}), \qquad \mu_0 = \text{permeability of free space}$$

$$\approx \frac{\mu_0}{4\pi} \left[\frac{3\hat{\vec{x}}\,(\hat{\vec{x}} \cdot \vec{\mu}) - \vec{\mu}}{r^3} + \mathcal{O}\left(\frac{1}{r^4}\right) \right], \qquad r = |\vec{x}|, \; \hat{\vec{x}} = \frac{\vec{x}}{r}$$

$$\vec{\mu} = \frac{1}{2} \int d^3y \, \vec{y} \times \vec{j}(\vec{y}) \quad \text{magnetic dipole moment}$$



No magnetic charges (no magnetic monopoles) $\rightarrow$ no "monopole contribution" $\sim 1/r^2$
Potential energy of magnetic moment in external mag. field: $V = -\vec{\mu} \cdot \vec{B}_{\text{ext}}(\vec{x})$
Torque on magnetic moment in (homogeneous) external mag. field: $\vec{M} = \vec{\mu} \times \vec{B}_{\text{ext}}$

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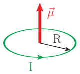

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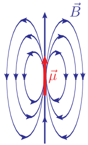

magnetic dipole field

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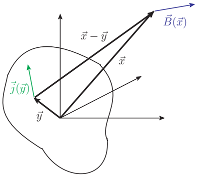

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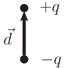
Electrostatics with static charge distribution $\rho(\vec{y})$ $[\mathrm{C/m^3}]$ (from Coulomb's law):

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3y\, \frac{\vec{x}-\vec{y}}{|\vec{x}-\vec{y}|^3}\,\rho(\vec{y}) \approx \frac{1}{4\pi\varepsilon_0}\left[\frac{\hat{\vec{x}}}{r^2}Q + \frac{3\hat{\vec{x}}\,(\hat{\vec{x}}\cdot\vec{p}) - \vec{p}}{r^3} + \mathcal{O}\left(\frac{1}{r^4}\right)\right]$$

$$Q = \int d^3y\, \rho(\vec{y}) \quad \text{total electric charge (electric "monopole moment")}$$

$$\vec{p} = \int d^3y\, \vec{y}\,\rho(\vec{y}) \quad \text{el. dipole moment, e.g. for 2 point charges: } \vec{p} = q\vec{d}$$



Magnetic moment of electron

- In quantum mechanics, an electron in a circular orbit around the nucleus has a magnetic (dipole) moment antiparallel to its quantized angular momentum $\vec{L}$:

$$\vec{\mu}_l = -g_l \frac{e}{2m_e} \vec{L} \quad \text{with} \quad g_l = 1, e > 0$$

- The study of the fine-structure of atomic spectra and the splitting of spectral lines in a weak external magnetic field (anomalous Zeeman effect) led Uhlenbeck & Goudsmit, 1925 to postulate the hypothesis of a "spinning electron" with an intrinsic quantized angular momentum $\vec{s}$ (spin).
- In analogy to orbital angular momentum they assumed that the electron had a corresponding intrinsic magnetic moment proportional to the spin:

$$\vec{\mu}_s = -g_s \frac{e}{2m_e} \vec{s}, \qquad g_s = \text{gyromagnetic factor (expected } g_s = 1)$$

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  Pure quantum mechanical effect. Electron pointlike, no rotating charged "ball" !
- However, from the experimental data on atomic spectra, it followed:

$$\begin{aligned} |\vec{s}| &= \hbar/2 \Rightarrow \text{electron has spin } 1/2 \\ |\vec{\mu}_s| &= \mu_B \equiv e\hbar/2m_e \quad \text{(Bohr magneton)} \\ &= 9.27 \times 10^{-24}\,\mathrm{Am}^2 = 0.579 \times 10^{-4}\,\mathrm{eV/T} \\ \Rightarrow g_s &= 2 \end{aligned}$$

- Dirac, 1928: equation for description of motion of electron in accordance with special theory of relativity, e.g. in an external (classical) magnetic field.
  $\Rightarrow g_{s,\text{Dirac}} = 2$
  This result lead to the quick acceptance of the Dirac theory. The prediction of antimatter (anti-electrons e^+ = positrons) came only later in 1930.

Anomalous magnetic moment

1947: hints of deviation from Dirac theory for electron in hyperfine structure of hydrogen. First direct measurement of g_s (Kusch & Foley, 1947/48):

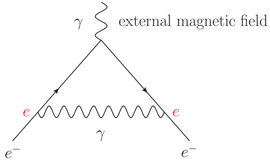
$$g_s = 2.00238 \pm 0.00010 \neq 2$$

Schwinger (1947/48): effect of virtual particles in quantum electrodynamics:

$$g_s = 2 + \frac{\alpha}{\pi} = 2.00232\ldots$$

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} \approx \frac{1}{137} = 0.00730\ldots$$

$\rightarrow$ agrees with experiment !



Virtual particles in loop

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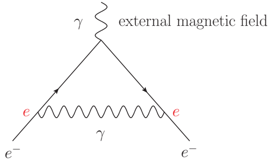


Virtual particles in loop

Anomalous magnetic moment a_e ("$g-2$") of electron (analogous for muon):

$$a_e = \frac{|\vec{\mu}_{s,e}| - |\vec{\mu}_{s,e,\mathrm{Dirac}}|}{|\vec{\mu}_{s,e,\mathrm{Dirac}}|} = \frac{g_{s,e}-2}{2} = \frac{\alpha}{2\pi} + \ldots = 0.001161\ldots + \ldots$$

After calculation of Lamb shift (Lamb & Retherford, 1947) by Bethe another important success for newly developed methods of quantum field theory.

a_e, a_μ: effect at the per-mille level. For comparison: $g_{\mathrm{s,proton}} = 5.59$, $g_{\mathrm{s,neutron}} = -3.83$ (neutral particle !) $\rightarrow$ big deviations from $g_{\mathrm{s,Dirac}} = 2$: early hints (1933, 1940) on substructure of proton and neutron. "Explained" by quark model (1960's).

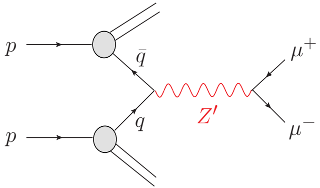
Tests of the Standard Model and search for New Physics

- Standard Model (SM) of particle physics very successful in precise description of a plenitude of experimental data, with a few exceptions ($3 - 4$ standard deviations).
- Some experimental facts (neutrino masses, baryon asymmetry in the universe, dark matter) and some theoretical arguments, which point to New Physics beyond the Standard Model.
- There are several indications that new particles (forces) should show up in the mass range $100\ \text{GeV} - 1\ \text{TeV}$.

Tests of the Standard Model and search for New Physics (continued)

Search for New Physics with two complementary approaches:

1. High Energy Physics:
   e.g. Large Hadron Collider (LHC) at CERN
   Direct production of new particles
   e.g. heavy Z' $\Rightarrow$ resonance peak in invariant mass distribution of $\mu^+\mu^-$ at $M_{Z'}$.



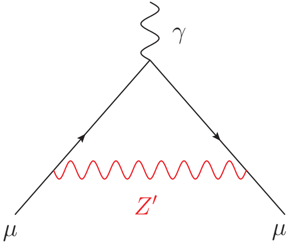
2. Precision physics:
   e.g. anomalous magnetic moments a_e, a_μ
   Indirect effects of virtual particles in quantum corrections
   $\Rightarrow$ Deviations from precise predictions in SM

$$\text{For } M_{Z'} \gg m_\ell : \quad a_\ell \sim \left(\frac{m_\ell}{M_{Z'}}\right)^2$$

$\Rightarrow$ a_μ by a factor $(m_\mu/m_e)^2 \sim 43000$ more sensitive to New Physics than a_e. Because of higher experimental precision for a_e: "only" factor 19.

Note: there are also non-decoupling contributions of heavy New Physics ! Another example: new light vector meson ("dark photon") with $M_V \sim (10 - 100)$ MeV.

a_e, a_μ allow to exclude some models of New Physics or to constrain their parameter space.



a_μ: why the 9th decimal place matters

$$a_\mu^{\text{exp}} = 0.00116592089(63) \quad [0.5 \text{ ppm} = 1 \text{ part in two millions}]$$

↕ 9th decimal place

$$a_\mu^{\text{SM}} = 0.00116591795(62)$$

$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (294 \pm 88) \times 10^{-11} \quad [3.3 \text{ standard deviations}]$$

⇒ Discrepancy in the 9th decimal place ! [8th decimal = 2 (rounded)]

a_μ^{exp}: Brookhaven $g-2$ experiment, Bennett et al., 2006; update: 2009
a_μ^{SM}: Jegerlehner & Nyffeler, 2009 (updated value here). Other groups: Davier et al. 2010; Hagiwara et al. 2011, also get about 3.5 standard deviations.

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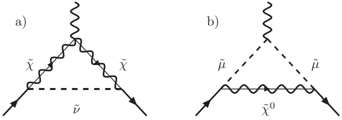
- Mistake in experiment ? Blind analysis for 1st publication in February 2001. Nevertheless, independent experimental confirmation would be very welcome.
- Mistake in theoretical calculation in Standard Model ?
  Underestimated theoretical uncertainty $\pm 62 \times 10^{-11}$?
  Problem: hadronic contributions from strong interactions (QCD).
- Contribution from New Physics ?

  e.g. Supersymmetry for large $\tan\beta$:

$$a_\mu^{\text{SUSY}} \approx 130 \times 10^{-11} \left(\frac{100 \text{ GeV}}{M_{\text{SUSY}}}\right)^2 \tan\beta$$



  (Czarnecki & Marciano, 2001)

  Explains discrepancy if $M_{\text{SUSY}} \approx (132 - 417)$ GeV $\quad (4 < \tan\beta < 40)$

Part 2

$g = 2$ from Dirac equation, $g - 2$ in Quantum Field Theory

$g = 2$ from Dirac equation (Itzykson + Zuber, QFT, Section 2.2.3)

Interaction of electron (Dirac Fermion) with external electromagnetic field $A_\mu(x)$ (minimal coupling prescription $\partial_\mu \to \partial_\mu - ieA_\mu, e > 0$):

$$\left[\gamma^\mu(i\partial_\mu + eA_\mu) - m\right]\psi(x) = 0$$

Non-relativistic limit

Write Dirac four-spinor in terms of two Pauli two-spinors and use Standard representation for Dirac matrices:

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

One obtains with $\vec{\pi} \equiv \vec{p} + e\vec{A}$:

$$\begin{aligned} i\frac{\partial\varphi}{\partial t} &= \vec{\sigma}\cdot\vec{\pi}\,\chi - eA^0\,\varphi + m\,\phi \\ i\frac{\partial\chi}{\partial t} &= \vec{\sigma}\cdot\vec{\pi}\,\varphi - eA^0\,\chi - m\,\chi \end{aligned}$$

In non-relativistic limit, the large energy m is the driving term. Introduce slowly varying functions of time Φ and X as follows: $\varphi = e^{-imt}\Phi$ and $\chi = e^{-imt}X$:

$$\begin{aligned} i\frac{\partial\Phi}{\partial t} &= \vec{\sigma}\cdot\vec{\pi}\,X - eA^0\,\Phi \\ i\frac{\partial X}{\partial t} &= \vec{\sigma}\cdot\vec{\pi}\,\Phi - eA^0\,X - 2m\,X \end{aligned}$$

If $eA^0 \ll 2m$ (weak field) and since X is slowly varying in time, one can solve 2nd equation algebraically: $X \simeq \frac{\vec{\sigma}\cdot\vec{\pi}}{2m}\Phi \ll \Phi$ (small and large components of Dirac spinor).

$g = 2$ from Dirac equation (continued)

Plugging solution for X back into first equation, one gets

$$i\frac{\partial \Phi}{\partial t} = \left[\frac{(\vec{\sigma}\cdot\vec{\pi})^2}{2m} - eA^0\right]\Phi$$

Using properties of Pauli matrices and taking into account that $\vec{\pi}$ contains the derivative operator and $\vec{A}(x)$, one has the identity

$$(\vec{\sigma}\cdot\vec{\pi})^2 = \sigma_i\sigma_j\pi^i\pi^j = \vec{\pi}^2 + \frac{1}{4}\left[\sigma_i,\sigma_j\right]\left[\pi^i,\pi^j\right] = \vec{\pi}^2 + e\vec{\sigma}\cdot\vec{B}$$

In this way one obtains the Pauli equation (generalization to spinors Φ of Schrödinger equation in electromagnetic field)

$$i\frac{\partial \Phi}{\partial t} = \mathbf{H}\Phi = \left[\frac{(\vec{p}+e\vec{A})^2}{2m} + \frac{e}{2m}\vec{\sigma}\cdot\vec{B} - eA^0\right]\Phi$$

Only spin-dependence through term $\vec{\sigma}\cdot\vec{B}$:

$$\mathbf{H}_{\text{magn}} = \frac{e}{2m}\vec{\sigma}\cdot\vec{B} \equiv -\vec{\mu}\cdot\vec{B}$$

with the magnetic moment defined as

$$\vec{\mu} = -\frac{e}{m}\frac{\vec{\sigma}}{2} = -2\left(\frac{e}{2m}\right)\vec{s}, \qquad \text{spin operator: } \vec{s} = \frac{\vec{\sigma}}{2}$$

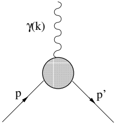
Gyromagnetic factor: $g_s = 2$

Anomalous magnetic moment in quantum field theory

Quantized spin 1/2 particle interacting with external, classical electromagnetic field

Interaction Lagrangian: $\mathcal{L}_{\rm int}^{\rm ext}(x) = -j^\mu(x) A_\mu^{\rm ext}(x)$ with conserved electromagnetic current: $j^\mu(x) = e\bar\psi(x)\gamma^\mu\psi(x)$

4 form factors in vertex function
(momentum transfer $k = p' - p$, not assuming parity or charge conjugation invariance)



$$\begin{aligned} &\equiv \quad i\langle p', s'|j^\mu(0)|p, s\rangle \\ &= \quad (-ie)\bar{\mathrm{u}}(p', s')\Bigg[\gamma^\mu \underbrace{F_1(k^2)}_{\text{Dirac}} + \frac{i\sigma^{\mu\nu}k_\nu}{2m}\underbrace{F_2(k^2)}_{\text{Pauli}} \\ &\qquad + \gamma^5\frac{\sigma^{\mu\nu}k_\nu}{2m}F_3(k^2) + \gamma^5\left(k^2\gamma^\mu - \not{k}\, k^\mu\right)F_4(k^2)\Bigg]\mathrm{u}(p, s) \end{aligned}$$

$\not{k} = \gamma^\mu k_\mu$. Real form factors for spacelike $k^2 \leq 0$. In the non-relativistic limit:

$$\begin{aligned} F_1(0) &= 1 && \text{(renormalization of charge } e\text{)} \\ \mu &= \frac{e}{2m}(F_1(0) + F_2(0)) && \text{(magnetic moment)} \\ a &= F_2(0) && \text{(anomalous magnetic moment)} \\ d &= -\frac{e}{2m}F_3(0) && \text{(electric dipole moment, violates P and CP)} \\ F_4(0) &= \text{anapole moment (violates P)} \end{aligned}$$

Anomalous magnetic moment in quantum field theory (continued)

Magnetic moment μ from interaction with weak, static external vector potential $A^\mu_{\rm ext}(x) = (0, \vec{A}_{\rm ext}(\vec{x}))$. Assume corresponding static magnetic field $\vec{B}(\vec{x})$ is slowly varying in space (essentially constant).

Dirac equation for spinors in momentum space: $(\not{p} - m)\mathrm{u}(p,s) = 0$. In rest frame of lepton and with Standard representation of Dirac matrices, one obtains two solutions (spin up, spin down):

$$\mathrm{u}((m,\vec{0}),s) = \sqrt{2m}\begin{pmatrix} \varphi^s \\ 0 \end{pmatrix}, \qquad \varphi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \varphi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Since $(\not{p} - m)(\not{p} + m) = 0$ for $p^2 = m^2$, we get general solution of Dirac equation:

$$\mathrm{u}(p,s) = \frac{(\not{p} + m)}{\sqrt{2m}\sqrt{p^0 + m}}\,\mathrm{u}((m,\vec{0}),s)$$

In non-relativistic limit $|\vec{p}|, |\vec{p}'| \ll m$ $(k_\mu \to 0)$ we get for vertex function (with spatial indices):

$$\begin{aligned}\bar{\mathrm{u}}(p',s')\gamma^i \mathrm{u}(p,s) &= \varphi^{s'\dagger}\left[(p^i + p'^i)\mathbf{1} - i\varepsilon^{ijk}k^j\sigma^k\right]\varphi^s = (p^i + p'^i)\delta_{s's} - i\varepsilon^{ijk}k^j\sigma^k_{s's} \\ \bar{\mathrm{u}}(p',s')\frac{i\sigma^{i\nu}k_\nu}{2m}\mathrm{u}(p,s) &= \varphi^{s'\dagger}\left[-i\varepsilon^{ijk}k^j\sigma^k\right]\varphi^s = -i\varepsilon^{ijk}k^j\sigma^k_{s's}\end{aligned}$$

where only the spin-dependent term σ^k will contribute to magnetic moment μ. Vertex function is multiplied with $A^i_{\rm ext}(k)$ and one obtains the Fourier transform of the magnetic field: $\sigma^k i\varepsilon^{ijk}k^j A^i_{\rm ext}(k) = \sigma^k B^k_{\rm ext}(k) = \vec{\sigma}\cdot\vec{B}_{\rm ext}(k)$.

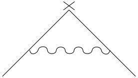
Some theoretical comments

- Anomalous magnetic moment is finite and calculable
  Corresponds to effective interaction Lagrangian of mass dimension 5:

$$\mathcal{L}_{\text{eff}}^{\text{AMM}} = -\frac{e_\ell a_\ell}{4m_\ell}\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)F_{\mu\nu}(x)$$

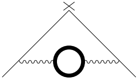
  $a_\ell = F_2(0)$ can be calculated unambiguously in renormalizable QFT, since there is no counterterm to absorb potential ultraviolet divergence.

- Anomalous magnetic moments are dimensionless
  To lowest order in perturbation theory in quantum electrodynamics (QED):



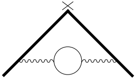
$$= a_e = a_\mu = \frac{\alpha}{2\pi}$$ [Schwinger 1947/48]

- Loops with different masses $\Rightarrow a_e \neq a_\mu$
  - Internal large masses decouple (often, but not always !):



$$= \left[\frac{1}{45}\left(\frac{m_e}{m_\mu}\right)^2 + \mathcal{O}\left(\frac{m_e^4}{m_\mu^4}\ln\frac{m_\mu}{m_e}\right)\right]\left(\frac{\alpha}{\pi}\right)^2$$

  - Internal small masses give rise to large log's of mass ratios:



$$= \left[\frac{1}{3}\ln\frac{m_\mu}{m_e} - \frac{25}{36} + \mathcal{O}\left(\frac{m_e}{m_\mu}\right)\right]\left(\frac{\alpha}{\pi}\right)^2$$

Electron $g - 2$

Electron $g-2$: Theory

Main contribution in Standard Model (SM) from mass-independent Feynman diagrams in QED with electrons in internal lines (perturbative series in α):

$$
\begin{aligned}
a_e^{\mathrm{SM}} \quad = \quad & \sum_{n=1}^{5} c_n \left(\frac{\alpha}{\pi}\right)^n \\
& +2.7478(2) \times 10^{-12} \quad [\text{Loops in QED with } \mu, \tau] \\
& +0.0297(5) \times 10^{-12} \quad [\text{weak interactions}] \\
& +1.682(20) \times 10^{-12} \quad [\text{strong interactions / hadrons}]
\end{aligned}
$$

The numbers are based on the paper by Aoyama et al. 2012.

QED: mass-independent contributions to a_e

- α: 1-loop, 1 Feynman diagram; Schwinger 1947/48:

  $c_1 = \frac{1}{2}$

- α^2: 2-loops, 7 Feynman diagrams; Petermann 1957, Sommerfield 1957:

  $c_2 = \frac{197}{144} + \frac{\pi^2}{12} - \frac{\pi^2}{2} \ln 2 + \frac{3}{4}\zeta(3) = -0.32847896557919378\ldots$

- α^3: 3-loops, 72 Feynman diagrams; ..., Laporta & Remiddi 1996:

$$
\begin{aligned}
c_3 &= \frac{28259}{5184} + \frac{17101}{810}\pi^2 - \frac{298}{9}\pi^2 \ln 2 + \frac{139}{18}\zeta(3) - \frac{239}{2160}\pi^4 \\
&\quad + \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\left\{\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{1}{24}\ln^4 2 - \frac{1}{24}\pi^2 \ln^2 2\right\} \\
&= 1.181241456587\ldots
\end{aligned}
$$

- α^4: 4-loops, 891 Feynman diagrams; Kinoshita et al. 1999, ..., Aoyama et al. 2008; 2012:

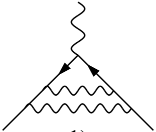
  $c_4 = -1.9106(20)$ (numerical evaluation)

- α^5: 5-loops, 12672 Feynman diagrams; Aoyama et al. 2005, ..., 2012:

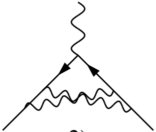
  $c_5 = 9.16(58)$ (numerical evaluation)

  Replaces earlier rough estimate $c_5 = 0.0 \pm 4.6$.
  New result removes biggest theoretical uncertainty in a_e !

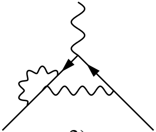
Mass-independent 2-loop Feynman diagrams in a_e



1)



2)



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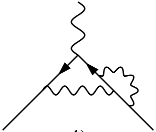

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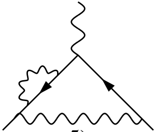
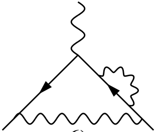


5)



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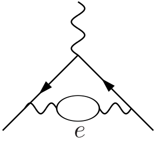
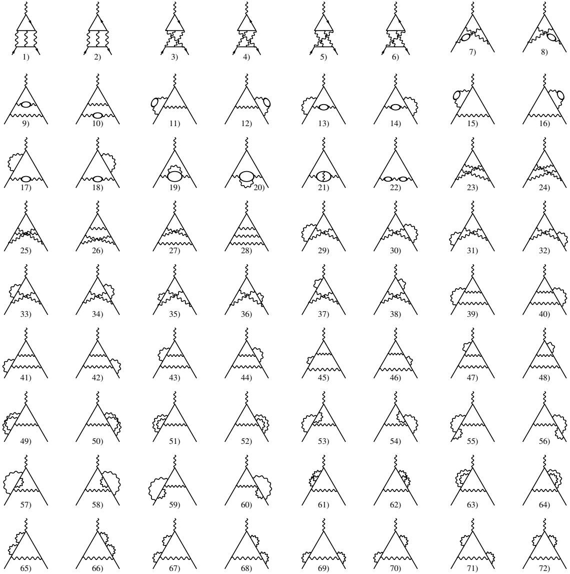


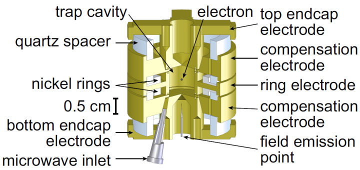
7)

Mass-independent 3-loop Feynman diagrams in a_e



Electron $g-2$: Experiment

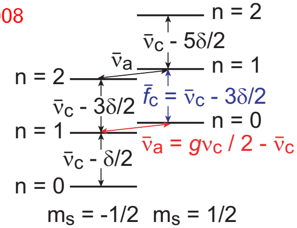
Latest experiment: Hanneke, Fogwell, Gabrielse, 2008



Cylindrical Penning trap for single electron

(1-electron quantum cyclotron)

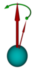
Source: Hanneke et al.



Cyclotron and spin precession levels of electron in Penning trap

Source: Hanneke et al.

$$\frac{g_e}{2} = \frac{\nu_s}{\nu_c} \simeq 1 + \frac{\bar{\nu}_a - \bar{\nu}_z^2/(2\bar{f}_c)}{\bar{f}_c + 3\delta/2 + \bar{\nu}_z^2/(2\bar{f}_c)} + \frac{\Delta g_{cav}}{2}$$



ν_s = spin precession frequency; $\nu_c, \bar{\nu}_c$ = cyclotron frequency: free electron, electron in Penning trap; $\delta/\nu_c = h\nu_c/(m_e c^2) \approx 10^{-9}$ = relativistic correction

4 quantities are measured precisely in experiment:

$\bar{f}_c = \bar{\nu}_c - \frac{3}{2}\delta \approx 149$ GHz; $\quad \bar{\nu}_a = \frac{g}{2}\nu_c - \bar{\nu}_c \approx 173$ MHz;

$\bar{\nu}_z \approx 200$ MHz = oscillation frequency in axial direction;

Δg_{cav} = corrections due to oscillation modes in cavity

$$\Rightarrow a_e^{\mathrm{exp}} = 0.00115965218073(28) \quad [0.24 \text{ ppb} \approx 1 \text{ part in 4 billions}]$$

Only the 12th decimal place uncertain ! (Kusch & Foley, 1947/48: 4% precision)

Precision in $g_e/2$ even 0.28 ppt $\approx$ 1 part in 4 trillions !

Determination of fine-structure constant α from $g-2$ of electron

- Recent measurement of α via recoil-velocity of Rubidium atoms in atom interferometer (Bouchendira et al. 2011):

$$\alpha^{-1}(\text{Rb}) = 137.035\ 999\ 037(91) \quad [0.66\text{ppb}]$$

This leads to (Aoyama et al. 2012):

$$a_e^{\text{SM}}(\text{Rb}) = 1\ 159\ 652\ 181.82\ \underbrace{(6)}_{c_4}\ \underbrace{(4)}_{c_5}\ \underbrace{(2)}_{\text{had}}\ \underbrace{(78)}_{\alpha(\text{Rb})}\ [78] \times 10^{-12} \quad [0.67\text{ppb}]$$

$$\Rightarrow a_e^{\text{exp}} - a_e^{\text{SM}}(\text{Rb}) = -1.09(0.83) \times 10^{-12} \quad [\text{Error from } \alpha(\text{Rb}) \text{ dominates !}]$$

$\rightarrow$ Test of QED !

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  $\longrightarrow$ Test of QED !

- Use a_e^{exp} to determine α from series expansion in QED (contributions from weak and strong interactions under control !). Assume: Standard Model "correct", no New Physics (Aoyama et al. 2012):

$$\alpha^{-1}(a_e) = 137.035\,999\,1657\ \underbrace{(68)}_{c_4}\ \underbrace{(46)}_{c_5}\ \underbrace{(24)}_{\text{had+EW}}\ \underbrace{(331)}_{a_e^{\text{exp}}}\ [342] \quad [0.25\text{ppb}]$$

  The uncertainty from theory has been improved by a factor 4.5 by Aoyama et al. 2012, the experimental uncertainty in a_e^{exp} is now the limiting factor.

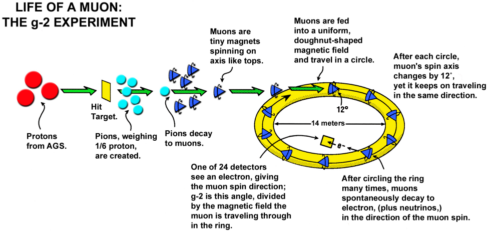
- Today the most precise determination of the fine-structure constant α, a fundamental parameter of the Standard Model.

Muon $g-2$

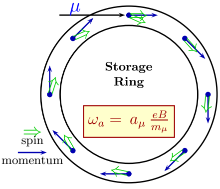
The Brookhaven Muon $g-2$ Experiment

The first measurements of the anomalous magnetic moment of the muon were performed in 1960 at CERN, $a_\mu^{\rm exp} = 0.00113(14)$ (Garwin et al.) [12% precision] and improved until 1979: $a_\mu^{\rm exp} = 0.0011659240(85)$ [7 ppm] (Bailey et al.)

In 1997, a new experiment started at the Brookhaven National Laboratory (BNL):



Source: BNL Muon $g-2$ homepage



actual precession × 2

Angular frequencies for cyclotron precession ω_c and spin precession ω_s:

$$\omega_c = \frac{eB}{m_\mu \gamma}\,, \quad \omega_s = \frac{eB}{m_\mu \gamma} + a_\mu \frac{eB}{m_\mu}\,, \quad \omega_a = a_\mu \frac{eB}{m_\mu}$$

$\gamma = 1/\sqrt{1-(v/c)^2}$. With an electric field to focus the muon beam one gets:

$$\vec{\omega}_a = \frac{e}{m_\mu}\left(a_\mu \vec{B} - \left[a_\mu - \frac{1}{\gamma^2 - 1}\right] \vec{v} \times \vec{E} \right)$$

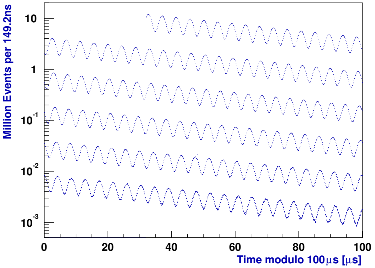
Term with $\vec{E}$ drops out, if $\gamma = \sqrt{1 + 1/a_\mu} = 29.3$: "magic γ" $\to p_\mu = 3.094$ GeV$/c$

The Brookhaven Muon $g-2$ Experiment: storage ring

Source: BNL Muon $g-2$ homepage

The Brookhaven Muon $g-2$ Experiment: determination of a_μ

Histogram with 3.6 billion decays of μ^-:



Bennett et al. 2006

$$N(t) = N_0(E)\ \exp\left(\frac{-t}{\gamma\tau_\mu}\right) \times [1 + A(E)\sin(\omega_a t + \phi(E))]$$

Exponential decay with mean lifetime: $\tau_{\mu,\text{lab}} = \gamma\tau_\mu = 64.378\mu s$ (in lab system).

Oscillations due to angular frequency $\omega_a = a_\mu eB/m_\mu$.

$$a_\mu = \frac{R}{\lambda - R} \text{ where } R = \frac{\omega_a}{\omega_p} \text{ and } \lambda = \frac{\mu_\mu}{\mu_p}$$

Brookhaven experiment measures ω_a and ω_p (spin precession frequency for proton). λ from hyperfine splitting of muonium ($\mu^+ e^-$) (external input).

Milestones in measurements of a_μ

Authors	Lab	Muon Anomaly	
Garwin et al. '60	CERN	0.001 13(14)	
Charpak et al. '61	CERN	0.001 145(22)	
Charpak et al. '62	CERN	0.001 162(5)	
Farley et al. '66	CERN	0.001 165(3)	
Bailey et al. '68	CERN	0.001 166 16(31)	
Bailey et al. '79	CERN	0.001 165 923 0(84)	
Brown et al. '00	BNL	0.001 165 919 1(59)	(μ^+)
Brown et al. '01	BNL	0.001 165 920 2(14)(6)	(μ^+)
Bennett et al. '02	BNL	0.001 165 920 4(7)(5)	(μ^+)
Bennett et al. '04	BNL	0.001 165 921 4(8)(3)	(μ^-)

World average experimental value (dominated by $g-2$ Collaboration at BNL, Bennett et al. '06 + CODATA 2008 value for $\lambda = \mu_\mu/\mu_p$):

$$a_\mu^{\text{exp}} = (116\,592\,089 \pm 63) \times 10^{-11} \quad [0.5\text{ppm}]$$

Goal of new planned $g-2$ experiments at Fermilab (partly recycled from BNL: moved ring magnet ! See pictures at: http://muon-g-2.fnal.gov/bigmove/) and J-PARC (completely new concept, not magic γ): $\delta a_\mu = 16 \times 10^{-11}$

Theory needs to match this precision !

For comparison: Electron (stable !) (Hanneke et al. '08):

$$a_e^{\text{exp}} = (1\,159\,652\,180.73 \pm 0.28) \times 10^{-12} \quad [0.24\text{ppb}]$$

Muon $g-2$: Theory

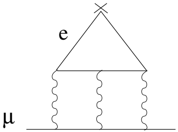
In Standard Model (SM):

$$a_\mu^{\text{SM}} = a_\mu^{\text{QED}} + a_\mu^{\text{weak}} + a_\mu^{\text{had}}$$

In contrast to a_e, here now the contributions from weak and strong interactions (hadrons) are relevant, since $a_\mu \sim (m_\mu/M)^2$.

QED contributions

- Diagrams with internal electron loops are enhanced.
- At 2-loops: vacuum polarization from electron loops enhanced by QED short-distance logarithm
- At 3-loops: light-by-light scattering from electron loops enhanced by QED infrared logarithm [Aldins et al. '69, '70; Laporta + Remiddi '93]



$+ \ldots$

$$a_\mu^{(3)}\Big|_{\text{lbyl}} = \left[\frac{2}{3}\pi^2 \ln\frac{m_\mu}{m_e} + \ldots\right]\left(\frac{\alpha}{\pi}\right)^3 = 20.947\ldots\left(\frac{\alpha}{\pi}\right)^3$$

- Loops with tau's suppressed (decoupling)

QED result up to 5 loops

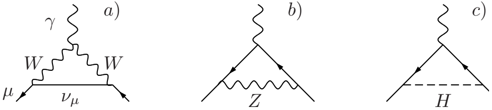
Include contributions from all leptons (Aoyama et al. 2012):

$$
\begin{aligned}
a_\mu^{\text{QED}} &= 0.5 \times \left(\frac{\alpha}{\pi}\right) + 0.765\,857\,425 \underbrace{(17)}_{m_\mu/m_{e,\tau}} \times \left(\frac{\alpha}{\pi}\right)^2 \\
&\quad + 24.050\,509\,96 \underbrace{(32)}_{m_\mu/m_{e,\tau}} \times \left(\frac{\alpha}{\pi}\right)^3 + 130.8796 \underbrace{(63)}_{\text{num. int.}} \times \left(\frac{\alpha}{\pi}\right)^4 \\
&\quad + 753.29 \underbrace{(1.04)}_{\text{num. int.}} \times \left(\frac{\alpha}{\pi}\right)^5 \\
&= 116\,584\,718.853 \underbrace{(9)}_{m_\mu/m_{e,\tau}} \underbrace{(19)}_{c_4} \underbrace{(7)}_{c_5} \underbrace{(29)}_{\alpha(a_e)} [36] \times 10^{-11}
\end{aligned}
$$

- Earlier evaluation of 5-loop contribution yielded $c_5 = 662(20)$ (Kinoshita, Nio 2006, numerical evaluation of 2958 diagrams, known or likely to be enhanced). New value is 4.5σ from this leading log estimate and 20 times more precise.
- Aoyama et al. 2012: What about the 6-loop term ? Leading contribution from light-by-light scattering with electron loop and insertions of vacuum-polarization loops of electrons into each photon line $\Rightarrow a_\mu^{\text{QED}}(\text{6-loops}) \sim 0.1 \times 10^{-11}$

Contributions from weak interaction

1-loop contributions [Jackiw, Weinberg, 1972; ...]:



$$a_\mu^{\text{weak, (1)}}(W) = \frac{\sqrt{2}G_\mu m_\mu^2}{16\pi^2}\frac{10}{3} + \mathcal{O}(m_\mu^2/M_W^2) = 388.70(0) \times 10^{-11}$$

$$a_\mu^{\text{weak, (1)}}(Z) = \frac{\sqrt{2}G_\mu m_\mu^2}{16\pi^2}\frac{(-1+4s_W^2)^2 - 5}{3} + \mathcal{O}(m_\mu^2/M_Z^2) = -193.89(2) \times 10^{-11}$$

Contribution from Higgs negligible: $a_\mu^{\text{weak, (1)}}(H) \leq 5 \times 10^{-14}$ for $m_H \geq 114$ GeV.

$$a_\mu^{\text{weak, (1)}} = (194.82 \pm 0.02) \times 10^{-11}$$

2-loop contributions (1678 diagrams):

$$a_\mu^{\text{weak, (2)}} = (-42.08 \pm 1.80) \times 10^{-11}, \quad \text{large since} \sim G_F m_\mu^2 \frac{\alpha}{\pi} \ln\frac{M_Z}{m_\mu}$$

Under control. Uncertainties from $m_H(100 - 300 \text{ GeV})$, 3-loop (RG): $\pm 1.5 \times 10^{-11}$, small hadronic uncertainty: $\pm 1.0 \times 10^{-11}$.

Total weak contribution:

$$a_\mu^{\text{weak}} = (153.2 \pm 1.8) \times 10^{-11}$$

Recent reanalysis by Gnendiger et al '13, using the now known Higgs mass, yields $a_\mu^{\text{weak}} = (153.6 \pm 1.0) \times 10^{-11}$ (error: 3-loop, hadronic).

Hadronic contributions to the muon $g-2$: largest source of error

- QCD: quarks bound by strong gluonic interactions into hadronic states
- In particular for the light quarks u, d, s $\longrightarrow$ cannot use perturbation theory !

Possible approaches to QCD at low energies:

- Lattice QCD: often still limited precision
- Effective quantum field theories with hadrons (ChPT): limited validity
- Simplifying hadronic models: model uncertainties not controllable
- Dispersion relations: extend validity of EFT's, reduce model dependence, often not all the needed input data available

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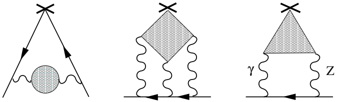
Different types of contributions to $g-2$:



(a) (b) (c)

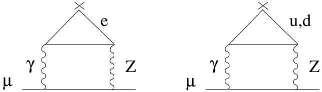
Light quark loop not well defined
$\rightarrow$ Hadronic "blob"

(a) Hadronic vacuum polarization (HVP) $\mathcal{O}(\alpha^2), \mathcal{O}(\alpha^3)$

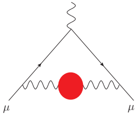
(b) Hadronic light-by-light scattering (HLbL) $\mathcal{O}(\alpha^3)$

(c) 2-loop electroweak contributions $\mathcal{O}(\alpha G_F m_\mu^2)$

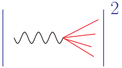
2-Loop EW
Small hadronic uncertainty from triangle diagrams.
Anomaly cancellation within each generation !
Cannot separate leptons and quarks !



Hadronic vacuum polarization

$a_\mu^{\text{HVP}} =$ 

Optical theorem (from unitarity; conservation of probability) for hadronic contribution $\rightarrow$ dispersion relation:

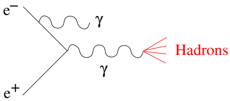
Im  $\sim$  $\sim$ $\sigma(e^+e^- \rightarrow \gamma^* \rightarrow \text{hadrons})$

$$a_\mu^{\text{HVP}} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{ds}{s}\, K(s)\, R(s), \qquad R(s) = \frac{\sigma(e^+e^- \rightarrow \gamma^* \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \gamma^* \rightarrow \mu^+\mu^-)}$$

[Bouchiat, Michel '61; Durand '62; Brodsky, de Rafael '68; Gourdin, de Rafael '69]

$K(s)$ slowly varying, positive function $\Rightarrow$ a_μ^{HVP} positive. Data for hadronic cross section σ at low center-of-mass energies $\sqrt{s}$ important due to factor $1/s$: $\sim 70\%$ from $\pi\pi$ [$\rho(770)$] channel, $\sim 90\%$ from energy region below 1.8 GeV.

Other method instead of energy scan: "Radiative return" at colliders with fixed center-of-mass energy (DAΦNE, B-Factories, BES) [Binner, Kühn, Melnikov '99; Czyż et al. '00-'03]

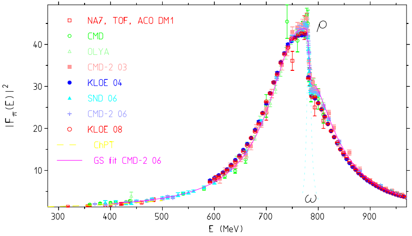


Measured hadronic cross-section

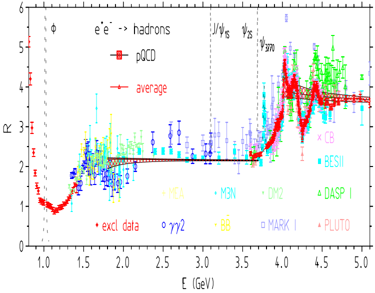
Pion form factor $|F_\pi(E)|^2$
($\pi\pi$-channel)

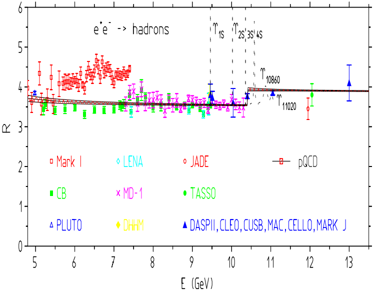
$$R(s) \quad = \quad \frac{1}{4}\left(1-\frac{4m_\pi^2}{s}\right)^{\frac{3}{2}} |F_\pi(s)|^2$$

$$(4m_\pi^2 < s < 9m_\pi^2)$$



R-ratio:





Jegerlehner + Nyffeler '09

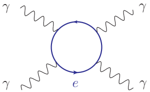
Hadronic vacuum polarization: some recent evaluations

Authors	Contribution to $a_\mu^{\rm HVP} \times 10^{11}$
Jegerlehner '08; JN '09 (e^+e^-)	6903.0 ± 52.6
Davier et al. '09 (e^+e^-) [+ τ]	6955 ± 41 [7053 ± 45]
Teubner et al. '09 (e^+e^-)	6894 ± 40
Davier et al. '10 (e^+e^-) [+ τ]	6923 ± 42 [7015 ± 47]
Jegerlehner + Szafron '11 (e^+e^-) [+ τ]	6907.5 ± 47.2 [6909.6 ± 46.5]
Hagiwara et al. '11 (e^+e^-)	6949.1 ± 42.7
Benayoun at al. '12 ($e^+e^- + \tau$: HLS improved)	6877.2 ± 46.3

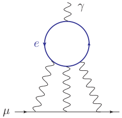
- Precision: $< 1\%$. Non-trivial because of radiative corrections (radiated photons).
- Even if values for $a_\mu^{\rm HVP}$ after integration agree quite well, the systematic differences of a few % in the shape of the spectral functions from different experiments (BABAR, CMD-2, KLOE, SND) indicate that we do not yet have a complete understanding.
- Use of τ data: additional sources of isospin violation ? Ghozzi + Jegerlehner '04; Benayoun et al. '08, '09; Wolfe + Maltman '09; Jegerlehner + Szafron '11 ($\rho - \gamma$-mixing), also included in HLS-approach by Benayoun et al. '12.
- Lattice QCD: Various groups are working on it (including at Mainz), precision at level of 5-10%, not yet competitive with phenomenological evaluations.

Hadronic light-by-light scattering in the muon $g-2$

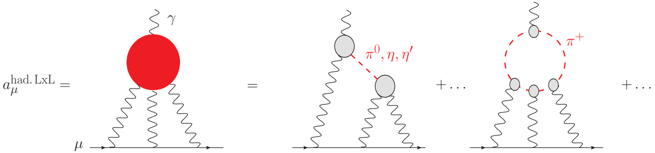
QED: light-by-light scattering at higher orders in perturbation series via lepton-loop:



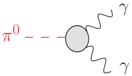
In muon $g-2$:
$\Rightarrow$



Hadronic light-by-light scattering in muon $g-2$ from strong interactions (QCD):



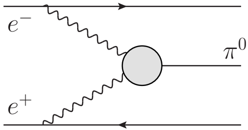
Coupling of photons to hadrons, e.g. π^0, via form factor:



View before 2014: in contrast to HVP, no direct relation to experimental data $\rightarrow$ size and even sign of contribution to a_μ unknown !
Approach: use hadronic model at low energies with exchanges and loops of resonances and some (dressed) "quark-loop" at high energies.
Problem: four-point function depends on several invariant momenta $\Rightarrow$ distinction between low and high energies not as easy as for two-point function in HVP.
Problem: mixed regions, where one loop momentum Q_1^2 is large and the other Q_2^2 is small and vice versa.

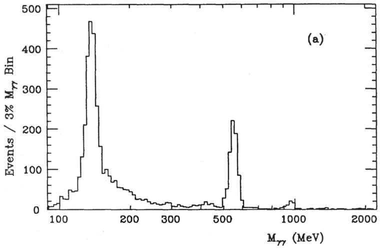
Experimental data on hadronic $\gamma\gamma \to \gamma\gamma$

In any case, it is a good idea to look at actual data. For instance, obtained by Crystal Ball detector '88 via $e^+e^- \to e^+e^-\gamma^*\gamma^* \to e^+e^-\pi^0$:



Feynman diagram from Colangelo et al., arXiv:1408.2517

Invariant $\gamma\gamma$ mass spectrum:



Three spikes from light pseudoscalars in reaction:

$$\gamma\gamma \to \pi^0, \eta, \eta' \to \gamma\gamma$$

for (almost) real photons.

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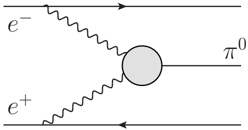

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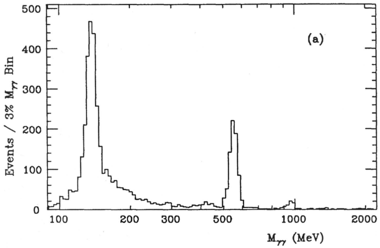

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for (almost) real photons.

New development in 2014: use of dispersion relations for a data driven approach to HLbL in muon $g-2$ from (still to be measured !) scattering of two off-shell photons:

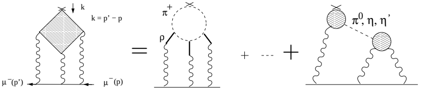
$$\begin{aligned} \gamma^*\gamma^* &\to \pi^0, \eta, \eta' \\ \gamma^*\gamma^* &\to \pi\pi \end{aligned}$$

(Colangelo et al.; Pauk, Vanderhaeghen).

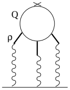
Current approach to HLbL scattering in $g - 2$

Classification of de Rafael '94

Chiral counting p^2 (from Chiral Perturbation Theory (ChPT)) and large-N_C counting as guideline to classify contributions (all higher orders in p^2 and N_C contribute):



+ ··· + Exchange of other resonances $(f_0, a_1, f_2 \ldots)$ +



+ ···

Chiral counting:	p^4	p^6	p^8	p^8
N_C-counting:	1	N_C	N_C	N_C
	pion-loop (dressed)	pseudoscalar exchanges		quark-loop (dressed)

Relevant scales in HLbL ($\langle VVVV \rangle$ with off-shell photons !): $0 - 2$ GeV, i.e. much larger than m_μ !

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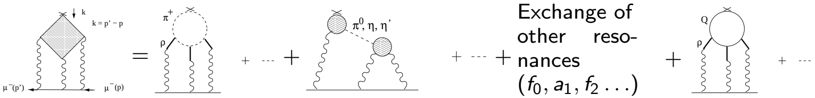

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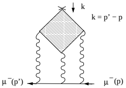
Relevant scales in HLbL ($\langle VVVV \rangle$ with off-shell photons !): $0-2$ GeV, i.e. much larger than m_μ !

Constrain models using experimental data (processes of hadrons with photons: decays, form factors, scattering) and theory (ChPT at low energies; short-distance constraints from pQCD / OPE at high momenta).

Issue: on-shell versus off-shell form factors. For instance pion-pole with on-shell pion form factors versus pion-exchange with off-shell pion form factors (in both cases with one or two off-shell photons).

Pseudoscalars: numerically dominant contribution (according to most models !).

HLbL scattering: Summary of selected results



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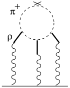

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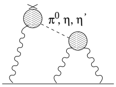

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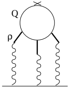

+ ···

Chiral counting:		p^4	p^6	p^8	p^8
N_C-counting:		1	N_C	N_C	N_C

Contribution to $a_\mu \times 10^{11}$:

	Total				
BPP:	+83 (32)	-19 (13)	+85 (13)	-4 (3) [f_0, a_1]	+21 (3)
HKS:	+90 (15)	-5 (8)	+83 (6)	+1.7 (1.7) [a_1]	+10 (11)
KN:	+80 (40)		+83 (12)		
MV:	+136 (25)	0 (10)	+114 (10)	+22 (5) [a_1]	0
2007:	+110 (40)				
PdRV:	+105 (26)	-19 (19)	+114 (13)	+8 (12) [f_0, a_1]	+2.3 [c-quark]
N,JN:	+116 (40)	-19 (13)	+99 (16)	+15 (7) [f_0, a_1]	+21 (3)
		ud.: -45	ud.: $+\infty$		ud.: +60

ud. = undressed, i.e. point vertices without form factors

BPP = Bijnens, Pallante, Prades '96, '02; HKS = Hayakawa, Kinoshita, Sanda '96, '98, '02; KN = Knecht, Nyffeler '02; MV = Melnikov, Vainshtein '04; 2007 = Bijnens, Prades; Miller, de Rafael, Roberts; PdRV = Prades, de Rafael, Vainshtein '09 (compilation; "Glasgow consensus"); N,JN = Nyffeler '09; Jegerlehner, Nyffeler '09 (compilation)

Total error estimates to large extent just guesses !

History: several changes in size and even sign due to errors since first evaluation by Calmet et al. '76. 2001: last sign change in dominant pseudoscalar contribution (KN '02).

Recall (in units of 10^{-11}): $\delta a_\mu(\text{HVP}) \approx 45$; $\delta a_\mu(\text{exp [BNL]}) = 63$; $\delta a_\mu(\text{future exp}) = 16$

Summary of recent developments

- Recent evaluations for pseudoscalars:

$$a_\mu^{\mathrm{HLbL};\pi^0} \sim (50-69)\times 10^{-11}$$
$$a_\mu^{\mathrm{HLbL;PS}} \sim (59-107)\times 10^{-11}$$

  Most evaluations agree at level of 15%, but some are quite different.
- New estimates for axial vectors (Pauk, Vanderhaeghen '14; Jegerlehner '14):

$$a_\mu^{\mathrm{HLbL;axial}} \sim (6-8)\times 10^{-11}$$

  Substantially smaller than in MV '04 !
- First estimate for tensor mesons (Pauk, Vanderhaeghen '14):

$$a_\mu^{\mathrm{HLbL;tensor}} = (1.1 \pm 0.1)\times 10^{-11}$$

- Open problem: Dressed pion-loop
  Potentially important effect from pion polarizability and a_1 resonance
  (Engel, Patel, Ramsey-Musolf '12; Engel '13; Engel, Ramsey-Musolf '13):

$$a_\mu^{\mathrm{HLbL};\pi-\mathrm{loop}} = -(11-71)\times 10^{-11}$$

  Maybe large negative contribution, in contrast to BPP '96, HKS '96.
- Open problem: Dressed quark-loop
  Dyson-Schwinger equation approach (Fischer, Goecke, Williams '11, '13):

$$a_\mu^{\mathrm{HLbL;quark-loop}} = 107\times 10^{-11} \qquad \text{(still incomplete !)}$$

  Large contribution, no damping seen, in contrast to BPP '96, HKS '96.

Data driven approach to HLbL using dispersion relations (DR)

Strategy: Split contributions to HLbL into two parts:

I: Data-driven evaluation using DR (hopefully numerically dominant):

(1) π^0, η, η' poles

(2) $\pi\pi$ intermediate state

II: Model dependent evaluation (hopefully numerically subdominant):

(1) Axial vectors (3π-intermediate state), ...

(2) Quark-loop, matching with pQCD

Error goals: Part I: 10% precision (data driven), Part II: 30% precision.
To achieve overall error of about 20% ($\delta a_\mu^{\mathrm{HLbL}} = 20 \times 10^{-11}$).

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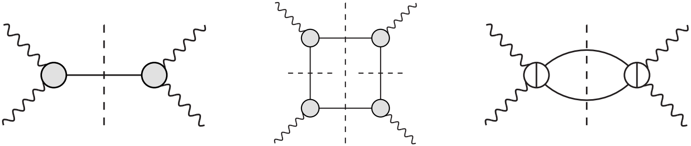
Colangelo et al., arXiv:1402.7081, arXiv:1408.2517:
Classify intermediate states in four-point function. Then project onto $g-2$.



$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0}_{\mu\nu\lambda\sigma} + \Pi^{\mathrm{FsQED}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\pi}_{\mu\nu\lambda\sigma} + \cdots$

$\Pi^{\pi^0}_{\mu\nu\lambda\sigma}$ = pion pole (similarly for η, η').

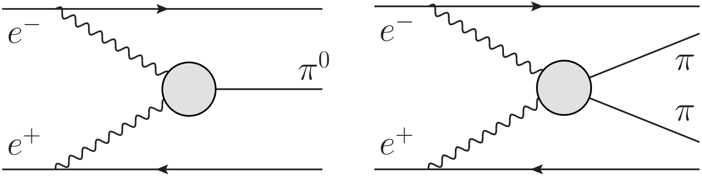
$\Pi^{\mathrm{FsQED}}_{\mu\nu\lambda\sigma}$ = scalar QED with vertices dressed by pion vector form factor F_π^V

$\Pi^{\pi\pi}_{\mu\nu\lambda\sigma}$ = remaining $\pi\pi$ contribution

Photon-photon processes in e^+e^- collisions

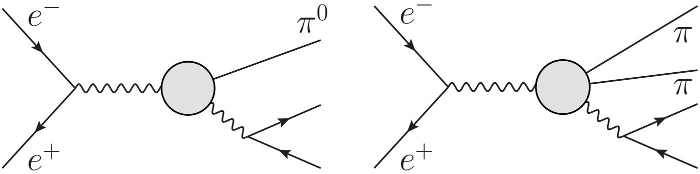
Feynman diagrams from Colangelo et al., arXiv:1408.2517

Space-like kinematics:



By tagging the outgoing leptons (single-tag, double-tag), one can infer the virtual (space-like) momenta $Q_i^2 = -q_i^2$ of the photons.
Left: process allows to measure pion-photon-photon transition form factor (TFF) $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q_1^2, Q_2^2)$.

Time-like kinematics:



Pion-pole contribution

Pion-pole contribution determined by measurable pion transition form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$ (on-shell pion, one or two off-shell photons). Analogously for η, η'-pole contributions.

Knecht, Nyffeler '02:

$$\begin{aligned} a_\mu^{\mathrm{HLbL};\pi^0} &= -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1+q_2)^2 [(p+q_1)^2 - m_\mu^2][(p-q_2)^2 - m_\mu^2]} \\ &\times \left[\frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, (q_1+q_2)^2) \ \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_2^2, 0)}{q_2^2 - m_\pi^2} \ T_1(q_1, q_2; p) \right. \\ &+ \left. \frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) \ \mathcal{F}_{\pi^0\gamma^*\gamma^*}((q_1+q_2)^2, 0)}{(q_1+q_2)^2 - m_\pi^2} \ T_2(q_1, q_2; p) \right] \end{aligned}$$

$$\begin{aligned} T_1(q_1, q_2; p) &= \frac{16}{3} (p\cdot q_1)(p\cdot q_2)(q_1\cdot q_2) - \frac{16}{3}(p\cdot q_2)^2 q_1^2 - \frac{8}{3}(p\cdot q_1)(q_1\cdot q_2) q_2^2 \\ &+ 8(p\cdot q_2) q_1^2 q_2^2 - \frac{16}{3}(p\cdot q_2)(q_1\cdot q_2)^2 + \frac{16}{3} m_\mu^2 q_1^2 q_2^2 - \frac{16}{3} m_\mu^2 (q_1\cdot q_2)^2 \\ T_2(q_1, q_2; p) &= \frac{16}{3} (p\cdot q_1)(p\cdot q_2)(q_1\cdot q_2) - \frac{16}{3}(p\cdot q_1)^2 q_2^2 + \frac{8}{3}(p\cdot q_1)(q_1\cdot q_2) q_2^2 \\ &+ \frac{8}{3}(p\cdot q_1) q_1^2 q_2^2 + \frac{8}{3} m_\mu^2 q_1^2 q_2^2 - \frac{8}{3} m_\mu^2 (q_1\cdot q_2)^2 \end{aligned}$$

where $p^2 = m_\mu^2$ and the external photon has now zero four-momentum (soft photon).

Currently, only single-virtual TFF $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q^2, 0)$ has been measured by CELLO, CLEO, BABAR, Belle. Analysis ongoing at BES, measurement planned at KLOE-2.

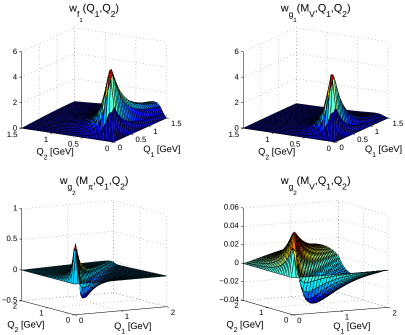
Relevant momentum regions in $a_\mu^{\mathrm{HLbL};\pi^0}$

- In Knecht, Nyffeler '02, a 2-dimensional integral representation for the pion-pole contribution was derived for a certain class of form factors (VMD-like). Schematically:

$$a_\mu^{\mathrm{HLbL};\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \sum_i w_i(Q_1, Q_2)\, f_i(Q_1, Q_2)$$

  with universal weight functions w_i. Dependence on form factors resides in the f_i.

- Plot of weight functions w_i from Knecht, Nyffeler '02:



- Relevant momentum regions around $0.25 - 1.25$ GeV. As long as form factors in different models lead to damping, expect comparable results for $a_\mu^{\mathrm{HLbL};\pi^0}$, at level of 20%.
- Jegerlehner, Nyffeler '09 derived 3-dimensional integral representation for general form factors (hyperspherical approach). Integration over $Q_1^2, Q_2^2, \cos\theta$, where $Q_1 \cdot Q_2 = |Q_1||Q_2|\cos\theta$.
- Idea recently taken up by Dorokhov et al. '12 (for scalars) and Bijnens, Zahiri Abyaneh '12-'14 (for all contributions).

$\pi\pi$ intermediate state

Colangelo et al., arXiv:1402.7081, arXiv:1408.2517

With pion vector form factor F_π^V from data, $\Pi_{\mu\nu\lambda\sigma}^{\text{FsQED}}$ is known to sufficient accuracy.

The remaining $\pi\pi$ contribution from $\Pi_{\mu\nu\lambda\sigma}^{\pi\pi}$ is then given by

$$a_\mu^{\pi\pi} = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{\sum_i I_i\left(s, q_1^2, q_2^2\right) T_i^{\pi\pi}\left(q_1, q_2; p\right)}{q_1^2 q_2^2 s Z_1 Z_2}$$

with $Z_1 = (p+q_1)^2 - m^2$, $Z_2 = (p-q_2)^2 - m^2$, $s = (q_1+q_2)^2$, $p^2 = m^2$ and known kinematical functions $T_i^{\pi\pi}(q_1, q_2; p)$, while information on the scattering amplitude on the cut is given by the dispersive integrals $I_i(s, q_1^2, q_2^2)$.

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For instance, for first S-wave:

$$I_1\left(s, q_1^2, q_2^2\right) = \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} \frac{ds'}{s'-s} \left[\left(\frac{1}{s'-s} - \frac{s' - q_1^2 - q_2^2}{\lambda\left(s', q_1^2, q_2^2\right)} \right) \operatorname{Im} h^0_{++,++}\left(s'; q_1^2, q_2^2; s, 0\right) + \frac{2\xi_1\xi_2}{\lambda\left(s', q_1^2, q_2^2\right)} \operatorname{Im} h^0_{00,++}\left(s'; q_1^2, q_2^2; s, 0\right) \right]$$

$\lambda(x,y,z) = x^2 + y^2 + z^2 - 2(xy + xz + yz)$: Källén function

ξ_i: normalization of longitudinal polarization vectors of off-shell photons

$h^J_{\lambda_1\lambda_2,\lambda_3\lambda_4}(s; q_1^2, q_2^2; q_3^2, q_4^2)$: partial-wave helicity amplitudes with angular momentum J for process $\gamma^*(q_1,\lambda_1)\gamma^*(q_2,\lambda_2) \to \gamma^*(q_3,\lambda_3)\gamma^*(q_4,\lambda_4)$.

Partial-wave unitarity relates imaginary parts in integrals I_i to helicity partial waves $h_{J,\lambda_1\lambda_2}(s; q_1^2, q_2^2)$ for $\gamma^*\gamma^* \to \pi\pi$, which have to be determined from experiment:

$$\operatorname{Im} h^J_{\lambda_1\lambda_2,\lambda_3\lambda_4}\left(s; q_1^2, q_2^2; q_3^2, q_4^2\right) = \frac{\sqrt{1 - 4M_\pi^2/s}}{16\pi} h_{J,\lambda_1\lambda_2}\left(s; q_1^2, q_2^2\right) h_{J,\lambda_3\lambda_4}\left(s; q_3^2, q_4^2\right)$$

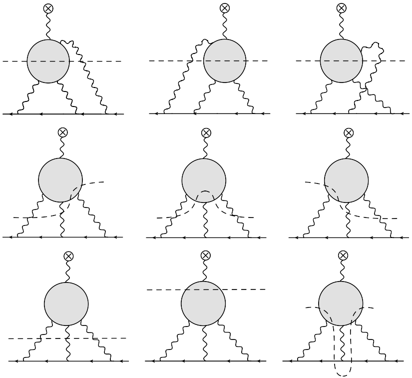
Another approach to HLbL using DR's

Pauk, Vanderhaeghen, arXiv:1403.7503, arXiv:1409.0819

Write DR directly for form factor $F_2(k^2)$:

$$F_2(0) = \frac{1}{2\pi i} \int_0^\infty \frac{\mathrm{d}k^2}{k^2} \mathrm{Disc}_{k^2} F_2(k^2)$$

Then study contributions from different intermediate states from the cuts in the unitarity diagrams related to measurable physical processes like $\gamma^*\gamma^* \to X$ and $\gamma^* \to \gamma X$:



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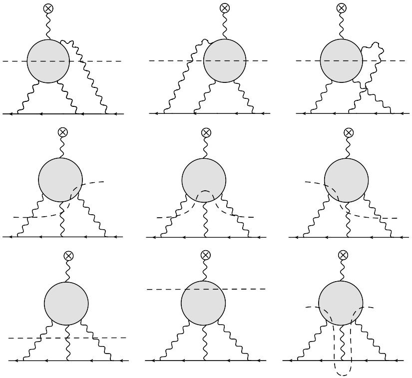
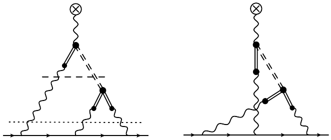


Pseudoscalar pole contribution

Considering the pseudoscalar intermediate state and VMD form factor ($\rho - \gamma$ mixing) it was shown that from dispersion relation one obtains precisely the result given by direct evaluation of two-loop integral in Knecht, Nyffeler '02 for the pseudoscalar pole.



Dashed: two-particle cut
Dotted: three-particle cut
Double-dashed: pseudoscalar pole
Double-solid: vector meson pole

Muon $g-2$: current status

Summary of SM contributions to a_μ (based on various recent sources):

- Leptonic QED contributions: $a_\mu^{\rm QED} = (116\,584\,718.853 \pm 0.036) \times 10^{-11}$
- Weak contributions: $a_\mu^{\rm weak} = (\underbrace{153.2}_{?} \pm \underbrace{1.8}_{?}) \times 10^{-11}$
- Hadronic contributions:
  - Vacuum Polarization:
    $a_\mu^{\rm HVP}(e^+e^-) = (\underbrace{6907.5}_{??} \pm \underbrace{47.2}_{??} - (100.3 \pm 2.2)) \times 10^{-11}$
  - Light-by-Light scattering: $a_\mu^{\rm HLbL} = (\underbrace{116 \pm 40}_{???}) \times 10^{-11}$
- Total SM contribution:

$$a_\mu^{\rm SM} = (116\,591\,795 \pm \underbrace{47}_{\rm HVP} \pm \underbrace{40}_{\rm HLbL} \pm \underbrace{1.8}_{\rm QED\,+\,EW} \quad [\pm 62]) \times 10^{-11}$$

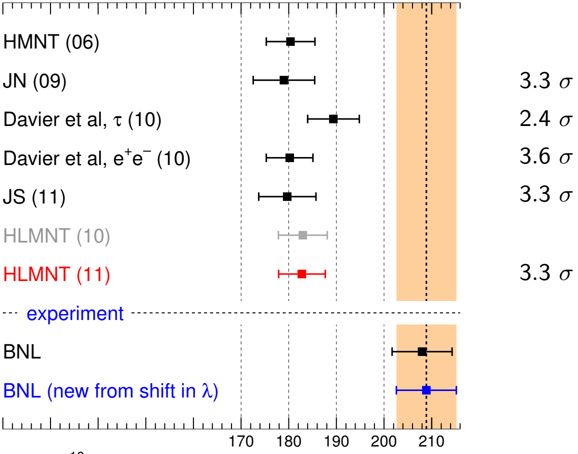
- "New" experimental value (shifted $+9.2 \times 10^{-11}$ due to new $\lambda = \mu_\mu/\mu_p$):

$$a_\mu^{\rm exp} = (116\,592\,089 \pm 63) \times 10^{-11}$$

$$\Rightarrow a_\mu^{\rm exp} - a_\mu^{\rm SM} = (294 \pm 88) \times 10^{-11} \qquad [3.3\,\sigma]$$

Sign of New Physics ? Hadronic uncertainties need to be better controlled in order to fully profit from future $g-2$ experiments with $\delta a_\mu = 16 \times 10^{-11}$.

Muon $g-2$: other recent evaluations



$a_\mu \times 10^{10} - 11659000$

Source: Hagiwara et al. '11. Note units of 10^{-10} !

Aoyama et al. '12: $a_\mu^{\rm exp} - a_\mu^{\rm SM} = (249 \pm 87) \times 10^{-11}$ [$2.9\ \sigma$]

Benayoun et al. '13: $a_\mu^{\rm exp} - a_\mu^{\rm SM} = 394 \times 10^{-11}$ [$4.9\ \sigma$]

New Physics contributions to the muon $g-2$

Define:

$$\Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (290 \pm 90) \times 10^{-11} \qquad \text{(Jegerlehner, Nyffeler '09)}$$

Absolute size of discrepancy is actually unexpectedly large, compared to weak contribution (although there is some cancellation there):

$$\begin{aligned} a_\mu^{\text{weak}} &= a_\mu^{\text{weak, (1)}}(W) + a_\mu^{\text{weak, (1)}}(Z) + a_\mu^{\text{weak, (2)}} \\ &= (389 - 194 - 42) \times 10^{-11} \\ &= 153 \times 10^{-11} \end{aligned}$$

Assume that New Physics contribution with $M_{\text{NP}} \gg m_\mu$ decouples:

$$a_\mu^{\text{NP}} = \mathcal{C} \frac{m_\mu^2}{M_{\text{NP}}^2}$$

where naturally $\mathcal{C} = \frac{\alpha}{\pi}$, like from a one-loop QED diagram, but with new particles. Typical New Physics scales required to satisfy $a_\mu^{\text{NP}} = \Delta a_\mu$:

$\mathcal{C}$	1	$\frac{\alpha}{\pi}$	$\left(\frac{\alpha}{\pi}\right)^2$
M_{NP}	$2.0^{+0.4}_{-0.3}$ TeV	100^{+21}_{-13} GeV	5^{+1}_{-1} GeV

Therefore, for New Physics model with particles in $250 - 300$ GeV mass range and electroweak-size couplings $\mathcal{O}(\alpha)$, we need some additional enhancement factor, like large $\tan\beta$ in the MSSM, to explain the discrepancy Δa_μ.

a_e, a_μ: Dark photon

In some dark matter scenarios, there is a relatively light, but massive "dark photon" A'_μ that couples to the SM through mixing with the photon:

$$\mathcal{L}_{\rm mix} = \frac{\varepsilon}{2} F^{\mu\nu} F'_{\mu\nu}$$

$\Rightarrow$ A'_μ couples to ordinary charged particles with strength $\varepsilon \cdot e$.
$\Rightarrow$ additional contribution of dark photon with mass m_V to the $g-2$ of a lepton (electron, muon) (Pospelov '09):

$$\begin{aligned} a_\ell^{\text{dark photon}} &= \frac{\alpha}{2\pi}\varepsilon^2 \int_0^1 dx \frac{2x(1-x)^2}{\left[(1-x)^2 + \frac{m_V^2}{m_\ell^2} x\right]} \\ &= \frac{\alpha}{2\pi}\varepsilon^2 \times \begin{cases} 1 & \text{for } m_\ell \gg m_V \\ \frac{2m_\ell^2}{3m_V^2} & \text{for } m_\ell \ll m_V \end{cases} \end{aligned}$$

For values $\varepsilon \sim (1-2) \times 10^{-3}$ and $m_V \sim (10-100)$ MeV, the dark photon could explain the discrepancy $\Delta a_\mu = 290 \times 10^{-11}$.

Various searches for the dark photon have been performed, are under way or are planned at BABAR, Jefferson Lab, KLOE, MAMI and other experiments. For a recent overview, see: *Dark Sectors and New, Light, Weakly-Coupled Particles* (Snowmass 2013), Essig et al., arXiv:1311.0029 [hep-ph].
Large parts of the parameter space in the (m_V, ε)-plane to explain the muon $g-2$ discrepancy have now been ruled out.

Conclusions

- Over many decades, the (anomalous) magnetic moments of the electron and the muon have played a crucial role in atomic and elementary particle physics.
- Experiment and Theory were thereby often going hand-in-hand, pushing each other to the limits.
- From a_e, a_μ we gained important insights into the structure of the fundamental interactions (quantum field theory).
- a_e: Test of QED, precise determination of fine-structure constant α.
  a_μ: Test of Standard Model, potential window to New Physics.

Outlook

- Current situation:

$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (294 \pm 88) \times 10^{-11} \qquad [3.3\ \sigma]$$

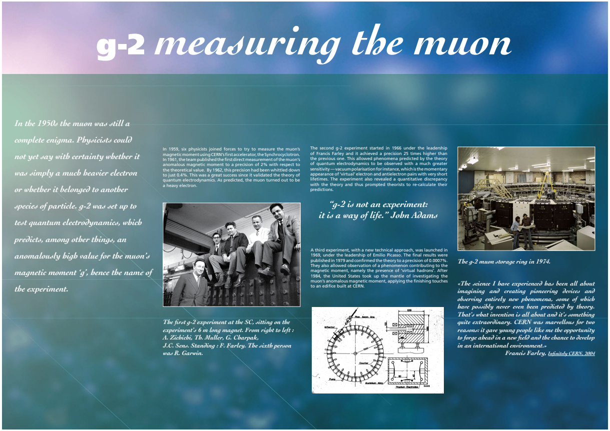
- Two new planned $g-2$ experiments at Fermilab and JPARC with goal of $\delta a_\mu^{\text{exp}} = 16 \times 10^{-11}$ (factor 4 improvement)
- Theory needs to match this precision !
- Hadronic vacuum polarization
  Ongoing and planned experiments on $\sigma(e^+e^- \to \text{hadrons})$ with a goal of $\delta a_\mu^{\text{HVP}} = (20 - 25) \times 10^{-11}$ (factor 2 improvement)
- Hadronic light-by-light scattering

$$a_\mu^{\text{HLbL}} = (105 \pm 26) \times 10^{-11} \qquad \text{(Prades, de Rafael, Vainshtein '09)}$$
$$a_\mu^{\text{HLbL}} = (116 \pm 40) \times 10^{-11} \qquad \text{(Nyffeler '09; Jegerlehner, Nyffeler '09)}$$

  Error estimates are mostly guesses ! Need a much better understanding of the complicated hadronic dynamics to get reliable error estimate of $\pm 20 \times 10^{-11}$.

  - Better theoretical models needed; more constraints from theory (ChPT, pQCD, OPE); close collaboration of theory and experiment to measure the relevant decays, form factors and cross-sections of various hadrons with photons.
  - Promising new data driven approach using dispersion relations for π^0, η, η' and $\pi\pi$. Still needed: data for scattering of off-shell photons.
  - Future: Lattice QCD.

And finally:



Source: CERN

"$g-2$ is not an experiment: it is a way of life."

John Adams (Head of the Proton Synchrotron at CERN (1954-61) and Director General of CERN (1960-1961))

This statement also applies to many theorists working on the $g-2$!

Backup

$g-2$ in QFT: Sketch of derivation (Weinberg, QFT, vol. 1, Section 10.6)

The magnetic moment μ for a particle of general spin j is defined by the matrix element of the interaction of the particle with a weak, static (time-independent), slowly varying in space (almost constant) magnetic field:

$$\langle p', s' | \mathbf{H}_{\rm ext} | p, s \rangle = -\frac{\mu}{j} \, (\vec{J}^{(j)})_{s's} \cdot \vec{B} \; \delta^{(3)}(\vec{p}\,' - \vec{p})$$

$(\vec{J}^{(j)})_{s's}$ is angular momentum matrix for spin j. For spin $\frac{1}{2}$: $\vec{J}^{(\frac{1}{2})} = \frac{\vec{\sigma}}{2}$

Matrix element = energy shift of incoming particle with momentum p with third component of spin s and outgoing particle with momentum p' with third component of spin s'.

Example: consider QED coupled to external, classical electromagnetic field $A_\mu^{\rm ext}(x)$. Leads to additional term in interaction Lagrangian:

$$\mathcal{L}_{\rm int}^{\rm ext}(x) = -j^\mu(x) A_\mu^{\rm ext}(x), \qquad j^\mu(x) = e\bar{\psi}(x)\gamma^\mu\psi(x)$$

Feynman rule in momentum space at vertex:  $= -ie\,\gamma^\mu\, A_\mu^{\rm ext}(k)$

(k: momentum transfer from external field; sometimes external field omitted)

Weak, static vector potential $A^\mu_{\rm ext}(x) = (0, \vec{A}_{\rm ext}(\vec{x})) \Rightarrow$ weak, static, magnetic field $\vec{B}(\vec{x})$. Assume that magnetic field is also slowly varying in space.

Interaction Hamiltonian: $\mathbf{H}_{\rm ext} = -\int d^3x\, \mathcal{L}_{\rm int}^{\rm ext}(x) = -\int d^3x\, \vec{j}(x) \cdot \vec{A}_{\rm ext}(\vec{x})$

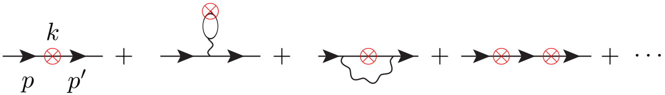
$g-2$ in QFT: Sketch of derivation (continued)

Movement of particle in presence of external field $A_\mu^{\rm ext}$ described by S-matrix:

$$\langle p', s', \mathrm{out} | p, s, \mathrm{in} \rangle = \langle p', s', \mathrm{in} | \, \mathbf{S} \, | p, s, \mathrm{in} \rangle$$

$$\mathbf{S} = \mathrm{T} \, \exp\left(i \int d^4x \, \mathcal{L}_{\rm int}(x) \right) \quad \text{(time-ordered product)}$$

Example QED: expand S-matrix in powers of α (from usual interaction term in QED) and powers of the external field $A_\mu^{\rm ext}(x)$ (assumed to be weak):



Very weak field: only one interaction with external field in $\mathcal{L}_{\rm int}^{\rm ext}(x) = -j^\mu(x) A_\mu^{\rm ext}(x)$ needs to be kept in expansion of S-matrix and we are led to study the matrix element of the current:

$$\langle p', s' | j^\mu(x) | p, s \rangle = \langle p', s' | e^{i\mathbf{P}\cdot x} j^\mu(0) e^{-i\mathbf{P}\cdot x} | p, s \rangle \quad \text{(translation invariance)}$$

$$= e^{i(p'-p)\cdot x} \langle p', s' | j^\mu(0) | p, s \rangle$$

Current conservation: $\partial_\mu j^\mu(x) = 0 \Rightarrow (p'-p)_\mu \langle p', s' | j^\mu(0) | p, s \rangle = 0$

$g-2$ in QFT: Sketch of derivation (continued)

Spin 1/2 particle

Using Lorentz invariance we must have:

$$i\langle p', s'|j^\mu(0)|p, s\rangle = (-ie)\bar{\mathrm{u}}(p', s')\Gamma^\mu(p', p)\mathrm{u}(p, s)$$

$\Gamma^\mu(p', p)$: four-vector 4×4 matrix function of p'_ν, p_ν and γ_ν.
$\bar{\mathrm{u}}(p', s')$,$\mathrm{u}(p, s)$ are the usual Dirac spinors in momentum space.
Expand $\Gamma^\mu(p', p)$ in the 16 covariant matrices $\mathbf{1}, \gamma_\rho, \sigma_{\rho\sigma} = (i/2)[\gamma_\rho, \gamma_\sigma], \gamma_5\gamma_\rho, \gamma_5$.
Linear combination of (assuming parity invariance for simplicity):

$$\begin{array}{rcl} \mathbf{1} & : & p^\mu, p'^\mu \\ \gamma_\rho & : & \gamma^\mu, p^\mu \not{p},\ p'^\mu \not{p},\ p^\mu \not{p}',\ p'^\mu \not{p}' \\ [\gamma_\rho, \gamma_\sigma] & : & [\gamma^\mu, \not{p}\,],\ [\gamma^\mu, \not{p}'\,],\ [\not{p}, \not{p}'\,]\, p^\mu,\ [\not{p}, \not{p}'\,]\, p'^\mu \\ \gamma_5\gamma_\rho & : & \gamma_5\gamma_\rho\epsilon^{\mu\rho\nu\sigma} p_\nu p'_\sigma \\ \gamma_5 & : & \text{none} \end{array}$$

with coefficients (form factors) which are functions of the only invariant k^2, where $k = p' - p$ and on-shell momenta $p'^2 = p^2 = m^2$. Use hermiticity of current $j^\dagger_\mu(x) = j_\mu(x)$ to get real form factors for $k^2 \le 0$.
Reduce number of terms by using: current conservation: $\partial_\mu j^\mu(x) = 0$, Dirac equation: $(\not{p} - m)\mathrm{u}(p, s) = 0$, $\bar{\mathrm{u}}(p', s')(\not{p}' - m) = 0$, various properties of Dirac matrices $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and of Dirac spinors, e.g. Gordon identities:

$$\begin{aligned} \bar{\mathrm{u}}(p', s')\gamma^\mu \mathrm{u}(p, s) &= \frac{1}{2m}\bar{\mathrm{u}}(p', s')\left[(p' + p)^\mu + i\sigma^{\mu\nu}(p' - p)_\nu\right]\mathrm{u}(p, s) \\ \bar{\mathrm{u}}(p', s')\gamma^\mu\gamma^5 \mathrm{u}(p, s) &= \frac{1}{2m}\bar{\mathrm{u}}(p', s')\left[(p' - p)^\mu\gamma^5 + i\sigma^{\mu\nu}(p' + p)_\nu\gamma^5\right]\mathrm{u}(p, s) \end{aligned}$$

Leads to 4 form factors $F_i(k^2)$ (not assuming parity or charge conjugation invariance).

In non-relativistic limit we finally get: $\mu = \frac{e}{2m}(F_1(0) + F_2(0))$ (magnetic moment),
$a = F_2(0)$ (anomalous magnetic moment)

$g-2$ unambigously calculable in renormalizable QFT

Anomalous magnetic moment corresponds to effective interaction Lagrangian

$$\mathcal{L}_{\mathrm{eff}}^{\mathrm{AMM}} = -\frac{e_\ell a_\ell}{4m_\ell}\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)F_{\mu\nu}(x)$$

with classical low-energy limit:

$$-\mathcal{L}_{\mathrm{eff}}^{\mathrm{AMM}} \Rightarrow \mathcal{H}_{\mathrm{m}} = \frac{e_\ell a_\ell}{2m_\ell}\vec{\sigma}\cdot\vec{B}$$

$\mathcal{L}_{\mathrm{eff}}^{\mathrm{AMM}}$ is mass dimension 5 operator ($\mathrm{dim}[\psi]=3/2, \mathrm{dim}[A_\mu]=1, \mathrm{dim}[\partial_\mu]=1$)
$\Rightarrow$ would spoil renormalizability of theory if it were present at tree-level.

$\Rightarrow a_\ell = F_2(0)$ can be calculated unambiguously in renormalizable QFT, since there is no counterterm to absorb potential ultraviolet (UV) divergence. a_ℓ also infrared (IR) finite, if there are no massless charged particles.

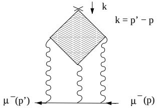
Note: Above derivation valid in QED. In Standard Model, the term $\delta\mathcal{L}_{\mathrm{eff}}^{\mathrm{AMM}}$ is not $SU(2)_{\mathrm{L}}$ gauge-invariant. Mixes left-handed and right-handed fields: $\bar{\psi}\sigma^{\mu\nu}\psi = \bar{\psi}_L\sigma^{\mu\nu}\psi_R + \bar{\psi}_R\sigma^{\mu\nu}\psi_L$. Need Higgs field Φ ($\mathrm{dim}[\Phi]=1$) to write a gauge-invariant dimension-6 operator:

$$\mathcal{L}_{\mathrm{eff}}^{\mathrm{AMM}} = -\frac{e_\ell a_\ell y_\ell}{4m_\ell^2}\left[\bar{L}_L(x)\Phi(x)\sigma^{\mu\nu}\ell_R(x)F_{\mu\nu}(x) + \mathrm{h.c.}\right]$$

$L_L = \begin{pmatrix} \nu_L \\ \ell_L \end{pmatrix}$: $SU(2)_L$ doublet, ℓ_R: $SU(2)_L$ singlet, $\Phi(x) = \frac{v+H(x)}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}$: Higgs doublet in unitary gauge, y_ℓ: Yukawa coupling for lepton: $m_\ell = y_\ell v/\sqrt{2}$

Hadronic light-by-light scattering in the muon $g-2$

$\mathcal{O}(\alpha^3)$ hadronic contribution to muon $g-2$: four-point function $\langle VVVV \rangle$ projected onto a_μ (external soft photon $k_\mu \to 0$).



Consider matrix element of light-quark electromagnetic current

$$j_\rho(x) = \frac{2}{3}(\bar{u}\gamma_\rho u)(x) - \frac{1}{3}(\bar{d}\gamma_\rho d)(x) - \frac{1}{3}(\bar{s}\gamma_\rho s)(x)$$

between muon states:

$$\langle \mu^-(p')|(ie)j_\rho(0)|\mu^-(p)\rangle = (-ie)\bar{\mathrm{u}}(p')\Gamma_\rho(p',p)\mathrm{u}(p)$$

$$= \int \frac{d^4q_1}{(2\pi)^4}\int \frac{d^4q_2}{(2\pi)^4} \frac{(-i)^3}{q_1^2\, q_2^2\,(q_1+q_2-k)^2} \frac{i}{(p'-q_1)^2-m^2} \frac{i}{(p'-q_1-q_2)^2-m^2}$$

$$\times(-ie)^3\,\bar{\mathrm{u}}(p')\gamma^\mu(\not{p}'-\not{q}_1+m)\gamma^\nu(\not{p}'-\not{q}_1-\not{q}_2+m)\gamma^\lambda \mathrm{u}(p)$$

$$\times(ie)^4\,\Pi_{\mu\nu\lambda\rho}(q_1,q_2,k-q_1-q_2)$$

with $k = p' - p$ and the fourth-rank light-quark hadronic tensor

$$\Pi_{\mu\nu\lambda\rho}(q_1,q_2,q_3) = \int d^4x_1 \int d^4x_2 \int d^4x_3\, e^{i(q_1\cdot x_1 + q_2\cdot x_2 + q_3\cdot x_3)} \langle\,\Omega\,|\mathrm{T}\{j_\mu(x_1)j_\nu(x_2)j_\lambda(x_3)j_\rho(0)\}|\,\Omega\,\rangle$$

Momentum conservation: $k = q_1 + q_2 + q_3$.

Projection onto $g-2$, Properties of $\Pi_{\mu\nu\lambda\rho}(q_1,q_2,q_3)$

Flavor diagonal current $j_\mu(x)$ is conserved and one has the Ward identities:

$$\{q_1^\mu; q_2^\nu; q_3^\lambda; k^\rho\}\, \Pi_{\mu\nu\lambda\rho}(q_1,q_2,q_3) \;=\; 0$$

$$\Rightarrow \Pi_{\mu\nu\lambda\rho}(q_1,q_2,k-q_1-q_2) = -k^\sigma \frac{\partial}{\partial k^\rho}\Pi_{\mu\nu\lambda\sigma}(q_1,q_2,k-q_1-q_2)$$

Defining $\Gamma_\rho(p',p) = k^\sigma \Gamma_{\rho\sigma}(p',p)$ one finally obtains (Aldins et al. '70):

$$a_\mu = F_2(0) \;=\; \frac{1}{48m}\,\mathrm{tr}\left((\not{p}+m)[\gamma^\rho,\gamma^\sigma](\not{p}+m)\Gamma_{\rho\sigma}(p,p)\right)$$

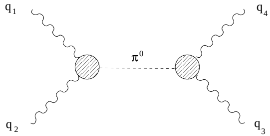
i.e. one can formally take the limit $k_\mu \to 0$ inside the loop integrals. Problem reduces to calculation of two-point function with zero-momentum insertion. One also gets better UV convergence properties of individual Feynman diagrams (fermion-loop).

Properties of $\Pi_{\mu\nu\lambda\rho}(q_1,q_2,q_3)$ (Bijnens et al. '95):

- In general 138 Lorentz structures. But only 32 contribute to $g-2$.
- Using Ward identities, there are 43 gauge invariant structures.
- Bose symmetry relates some of them.
- All depend on $q_1^2, q_2^2, q_3^2, q_i\cdot q_j$, but before taking derivative and $k_\mu \to 0$, also on $k^2, k\cdot q_i$.
- Compare with HVP: one function, one variable.

Pion-pole versus pion-exchange in $a_\mu^{\mathrm{HLbL};\pi^0}$

- To uniquely identify contribution of exchanged neutral pion π^0 in Green's function $\langle VVVV \rangle$, we need to pick out pion-pole:

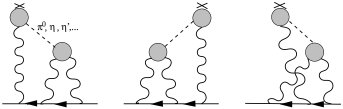


+ crossed diagrams

$$\lim_{(q_1+q_2)^2 \to m_\pi^2} ((q_1+q_2)^2 - m_\pi^2) \langle VVVV \rangle$$

Residue of pole: on-shell vertex function $\langle 0 | VV | \pi \rangle$
$\to$ on-shell form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$

- But in contribution to muon $g-2$, we evaluate Feynman diagrams, integrating over photon momenta with exchanged off-shell pions. For all the pseudoscalars:



Shaded blobs represent off-shell form factor $\mathcal{F}_{\mathrm{PS}^*\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2)$ where $\mathrm{PS} = \pi^0, \eta, \eta', \pi^{0\prime}, \ldots$

Off-shell form factors are either inserted "by hand" starting from constant, pointlike Wess-Zumino-Witten (WZW) form factor or using e.g. some resonance Lagrangian. Contribution with off-shell form factors is model dependent.

- Within a dispersive framework the pole contribution can be uniquely defined from imaginary part of Feynman diagram (similarly for multi-particle intermediate states like $\pi\pi$), but this represents only part of the contribution.

HLbL scattering: Summary of selected results

Some results for the various contributions to $a_\mu^{\mathrm{HLbL}} \times 10^{11}$:

Contribution	BPP	HKS, HK	KN	MV	BP, MdRR	PdRV	N, JN
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	–	114 ± 13	99 ± 16
axial vectors	2.5 ± 1.0	1.7 ± 1.7	–	22 ± 5	–	15 ± 10	22 ± 5
scalars	-6.8 ± 2.0	–	–	–	–	-7 ± 7	-7 ± 2
π, K loops	-19 ± 13	-4.5 ± 8.1	–	–	–	-19 ± 19	-19 ± 13
π, K loops +subl. N_C	–	–	–	0 ± 10	–	–	–
quark loops	21 ± 3	9.7 ± 11.1	–	–	–	2.3 (c-quark)	21 ± 3
Total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	110 ± 40	105 ± 26	116 ± 39

BPP = Bijnens, Pallante, Prades '95, '96, '02; HKS = Hayakawa, Kinoshita, Sanda '95, '96; HK = Hayakawa, Kinoshita '98, '02; KN = Knecht, Nyffeler '02; MV = Melnikov, Vainshtein '04; BP = Bijnens, Prades '07; MdRR = Miller, de Rafael, Roberts '07; PdRV = Prades, de Rafael, Vainshtein '09; N = Nyffeler '09, JN = Jegerlehner, Nyffeler '09

- Pseudoscalar-exchanges dominate numerically. Other contributions not negligible. Cancellation between π, K-loops and quark loops !
- PdRV: Analyzed results obtained by different groups with various models and suggested new estimates for some contributions (shifted central values, enlarged errors). Do not consider dressed light quark loops as separate contribution ! Assume it is already taken into account by using short-distance constraint of MV '04 on pseudoscalar-pole contribution. Added all errors in quadrature !
- N, JN: New evaluation of pseudoscalar exchange contribution imposing new short-distance constraint on off-shell form factors. Took over most values from BPP, except axial vectors from MV. Added all errors linearly.

Form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$ and transition form factor $F(Q^2)$

- Form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$ between an on-shell pion and two off-shell photons:

$$i \int d^4x\, e^{iq_1 \cdot x} \langle 0 | T\{j_\mu(x) j_\nu(0)\} | \pi^0(q_1 + q_2)\rangle = \varepsilon_{\mu\nu\alpha\beta}\, q_1^\alpha\, q_2^\beta\, \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$$

$$j_\mu(x) = (\overline{\psi}\hat{Q}\gamma_\mu\psi)(x), \quad \psi \equiv \begin{pmatrix} u \\ d \\ s \end{pmatrix}, \quad \hat{Q} = \mathrm{diag}(2, -1, -1)/3$$

(light quark part of electromagnetic current)

Bose symmetry: $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_2^2, q_1^2)$

Form factor for real photons is related to $\pi^0 \to \gamma\gamma$ decay width:

$$\mathcal{F}^2_{\pi^0\gamma^*\gamma^*}(q_1^2 = 0, q_2^2 = 0) = \frac{4}{\pi\alpha^2 m_\pi^3} \Gamma_{\pi^0\to\gamma\gamma}$$

Often normalization with chiral anomaly is used:

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0, 0) = -\frac{1}{4\pi^2 F_\pi}$$

- Pion-photon transition form factor:

$$F(Q^2) \equiv \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q^2, q_2^2 = 0), \qquad Q^2 \equiv -q_1^2$$

Note that $q_2^2 = 0$, but $\vec{q}_2 \neq \vec{0}$ for on-shell photon !

Impact of form factor measurements: example KLOE-2

On the possibility to measure the $\pi^0 \to \gamma\gamma$ decay width and the $\gamma^*\gamma \to \pi^0$ transition form factor with the KLOE-2 experiment

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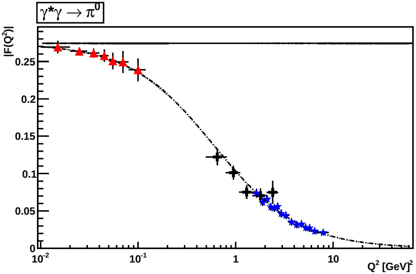

Simulation of KLOE-2 measurement of TFF $F(Q^2)$ (red triangles). MC program EKHARA 2.0 (Czyż, Ivashyn '11) and detailed detector simulation.
Solid line: $F(0)$ given by chiral anomaly (WZW).
Dashed line: form factor according to on-shell LMD+V model (Knecht, Nyffeler '01).
CELLO (black crosses) and CLEO (blue stars) data at higher Q^2.

Within 1 year of data taking, collecting 5 fb^{-1}, KLOE-2 will be able to measure:

- $\Gamma_{\pi^0 \to \gamma\gamma}$ to 1% statistical precision.
- $\gamma^*\gamma \to \pi^0$ transition form factor $F(Q^2)$ in the region of very low, space-like momenta $0.01 \text{ GeV}^2 \leq Q^2 \leq 0.1 \text{ GeV}^2$ with a statistical precision of less than 6% in each bin.
  KLOE-2 can (almost) directly measure slope of form factor at origin (note: logarithmic scale in Q^2 in plot !).

Impact of form factor measurements: example KLOE-2 (continued)

- Error in $a_\mu^{\mathrm{HLbL};\pi^0}$ related to model parameters determined by $\Gamma_{\pi^0\to\gamma\gamma}$ (normalization of FF; not taken into account in most papers) and $F(Q^2)$ will be reduced as follows:
  - $\delta a_\mu^{\mathrm{HLbL};\pi^0} \approx 4 \times 10^{-11}$ (with current data for $F(Q^2) + \Gamma^{\mathrm{PDG}}_{\pi^0\to\gamma\gamma}$)
  - $\delta a_\mu^{\mathrm{HLbL};\pi^0} \approx 2 \times 10^{-11}$ ($+ \Gamma^{\mathrm{PrimEx}}_{\pi^0\to\gamma\gamma}$)
  - $\delta a_\mu^{\mathrm{HLbL};\pi^0} \approx (0.7 - 1.1) \times 10^{-11}$ (+ KLOE-2 data)
- Note that error does not account for other potential uncertainties in $a_\mu^{\mathrm{HLbL};\pi^0}$, e.g. related to choice of model, 2nd off-shell photon, off-shellness of pion.
- Simple models with few parameters, like VMD (two parameters: F_π, M_V), which are completely determined by the data on $\Gamma_{\pi^0\to\gamma\gamma}$ and $F(Q^2)$, can lead to very small errors in $a_\mu^{\mathrm{HLbL};\pi^0}$. For illustration:
  $a_{\mu;\mathrm{VMD}}^{\mathrm{HLbL};\pi^0} = (57.3 \pm 1.1) \times 10^{-11}$
  $a_{\mu;\mathrm{LMD+V}}^{\mathrm{HLbL};\pi^0} = (72 \pm 12) \times 10^{-11}$ (off-shell LMD+V FF, including all errors)
- But this might be misleading ! VMD and LMD+V give equally good fits to transition form factor $F(Q^2)$, but differ in doubly-off shell transition form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$. Results for $a_\mu^{\mathrm{HLbL};\pi^0}$ differ by about 20% ! Reason: VMD form factor has wrong high-energy behavior $\Rightarrow$ too strong damping in $a_{\mu;\mathrm{VMD}}^{\mathrm{HLbL};\pi^0}$:
  $\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(q^2, q^2) \sim 1/q^4$, for large q^2, i.e. falls off too fast compared to OPE prediction $\mathcal{F}^{\mathrm{OPE}}_{\pi^0\gamma^*\gamma^*}(q^2, q^2) \sim 1/q^2$ which is fulfilled by LMD+V.
  $\Rightarrow$ Dispersive approach to not rely (or less) on models.