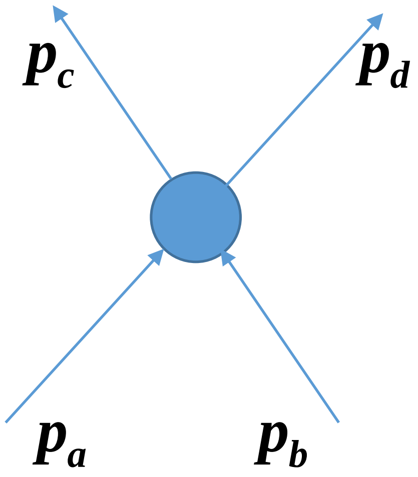
# General properties of scattering matrix- S- matrix

Scattering process $a + b \rightarrow c + d$ is represented by diagram




Out of four 4 - momenta we can define three lorentz invariant quantities :

$$s = (p_a + p_b)^2 = w^2$$

$$t = (p_a - p_c)^2$$

$$u = (p_a + p_d)^2$$

$$s + t + u = \Sigma$$

$$\Sigma = m_a^2 + m_a^2 + m_c^2 + m_d^2$$

In scattering theory it is shown that one has to consider

additional processes (channels) obtained by so called crossing operation :

$$a + \bar{c} \rightarrow \bar{b} + d$$

$$a + \bar{d} \rightarrow c + \bar{b}$$

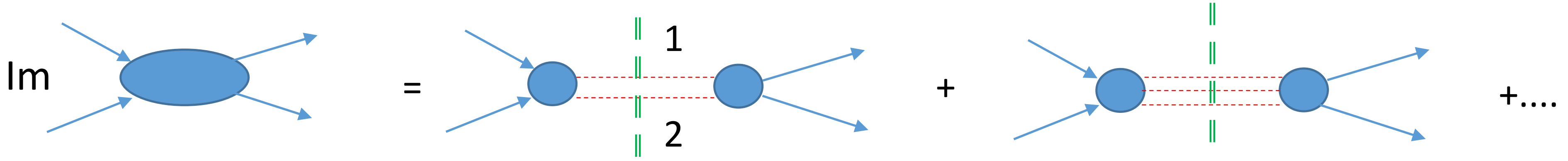
Scattering is described by a scattering matrix - S matrix :

$$\langle f|S|i\rangle = \delta_{if} - (2\pi)^4 i \cdot \delta(p_f - p_i)\langle f|T|i\rangle$$

$S$ – matrix is unitary : $S^+S = 1$, what gives : $T\text{-}T^+ = iT^+T$

$$\langle f|T - T^+|i\rangle = \langle f|T|i\rangle - \langle f|T^+|i\rangle = (2\pi)^4 \cdot \sum_n \langle f|T^+|n\rangle\langle n|T|i\rangle \cdot \delta(p_i - p_n)$$

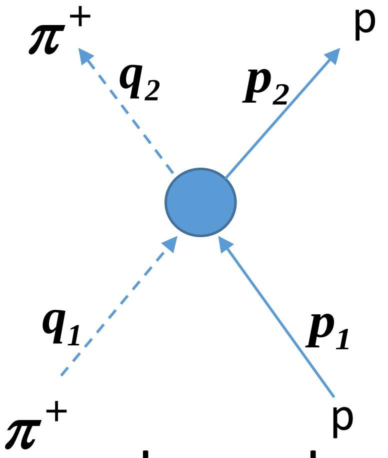
$$2\, Im\langle f|T|i\rangle = (2\pi)^4 \cdot \sum_n \langle f|T^+|n\rangle\langle n|T|i\rangle \cdot \delta(p_i - p_n) - \text{unitarity relation}$$




$$Im\langle f|T|i\rangle \neq 0 \quad if\ s = s_{th} \leq (m_1 + m_2)^2,$$

where $m_1$ and $m_2$ are masses of the lowest intermediate state.

# $\pi\ N$ scattering



$s$ channel

$$s=(q_1+p_1)^2=w^2$$

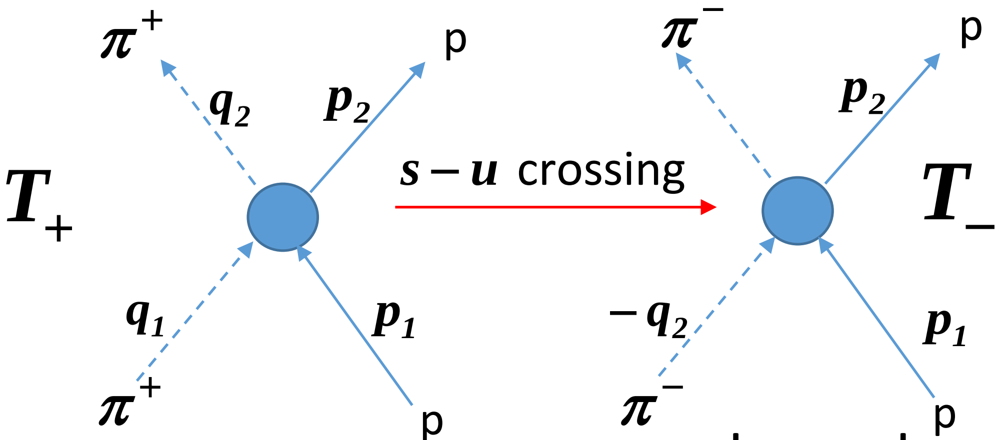
$$t=(q_1-q_2)^2 \qquad s+t+u=\Sigma=2m^2+2m_\pi^2$$

$$u=(q_1-p_2)$$

In $\pi N$ scattering, transition matrix T is given in terms of two invariant amplitudes A(s,t,u) and B(s,t,u):

$T^I(s,t,u)=-A^I(s,t,u)+i(q_1+q_2)_\mu\gamma^\mu B^I$ , where I stands for isospin, $\mathrm{I}=\frac{1}{2},\frac{3}{2}$

Using crossing operation, we obtain another two channels:



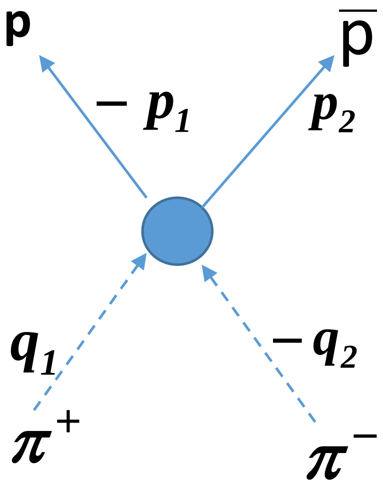
u channel

$$s=(p_1-q_2)^2$$

$$t=(q_1-q_2)^2 \quad \text{unchanged}$$

$$u=(-q_2-p_2)^2=(p_1+q_1)^2=w_u^2$$

Similarly, by ***s-t*** crossing one obtains a process



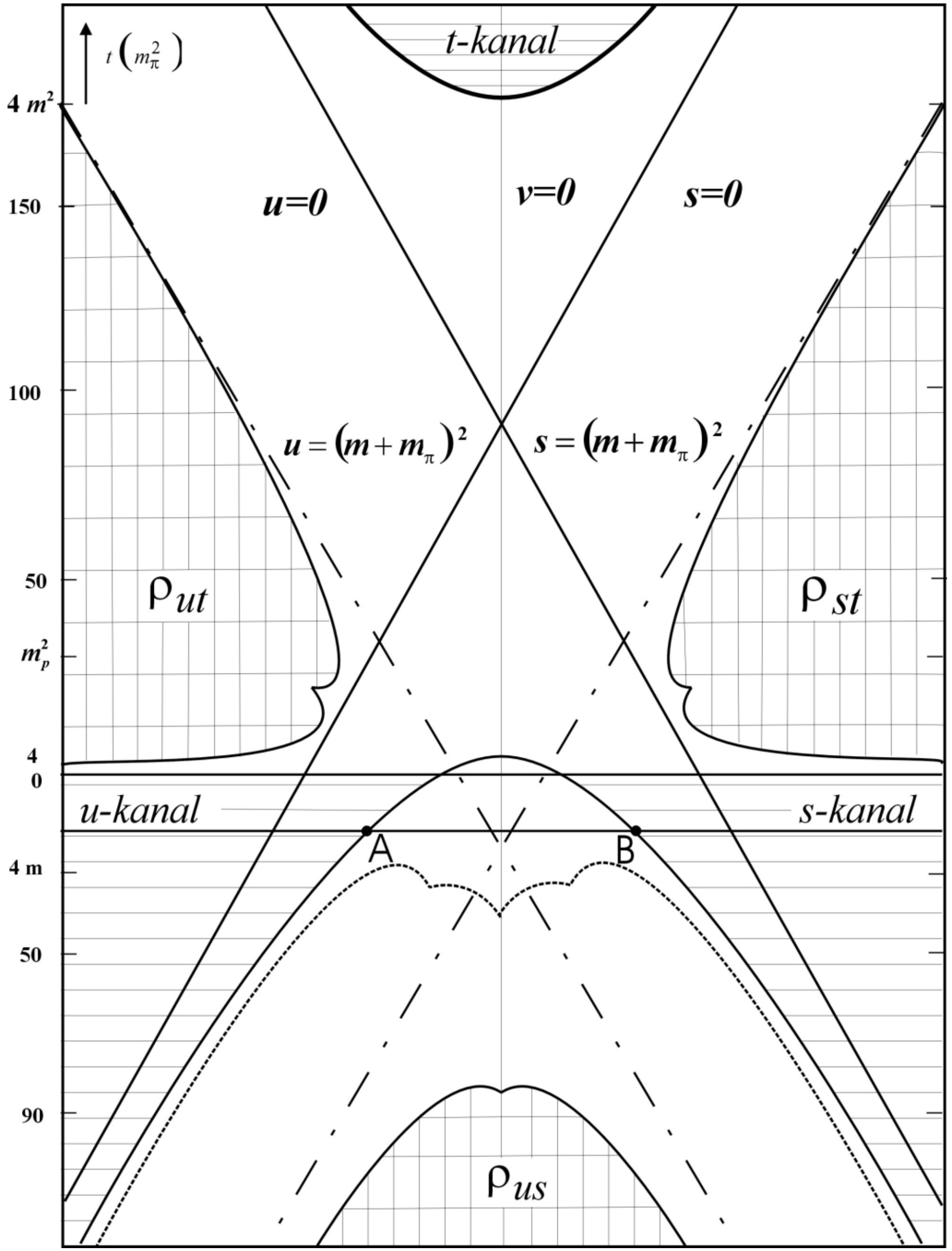

$$s = (q_1 - q_2)^2$$
$$t = (q_1 + q_2)^2 = w_t^2$$
$$u = (q_1 - p_2)^2$$

## Crossing symmetry.

In S-matrix theory it is shown:

Processes obtained by crossing operation are described by the same T matrix (Invariant amplitudes):

$$A_+(s,t,u) = A\_(u,t,s)$$
$$B_+(s,t,u) = B\_(u,t,s$$



t-kanal
$t\,(m_\pi^2)$
$4\ m^2$
150
100
50
$m_p^2$
4
0
4 m
50
90
u=0
v=0
s=0
$u=(m+m_\pi)^2$
$s=(m+m_\pi)^2$
$\rho_{ut}$
$\rho_{st}$
u-kanal
s-kanal
A
B
$\rho_{us}$

***Mandelstam Hypothesis applied to $\pi N$ scattering***

The $\boldsymbol{\pi N}$ Invariant amplitudes are analytic functions of complex variables $\boldsymbol{s,t}$ and $\boldsymbol{u}$.

All singularities occure at the real values of variables $\boldsymbol{s, t, u}$.

Singularities are these derived from unitarity in $\boldsymbol{s,t}$ and $\boldsymbol{u}$ channels:

i) Cats starting at energy squared corresponding to mass squared of the lowest intermediate state in unitarity relation for a given channel.

For A amplitudes:

$(m+m_\pi)^2 \le s < \infty$ s-channel cut

$(m+m_\pi)^2 \le u < \infty$ u-channel cut

$4m_\pi^2 \le t < \infty$ t-channel cut

In addition to cuts present in amplitude A, invariant amplitude B has two poles

at $u = m^2$ and at $s = m^2$.

ii) Physical amplitudes are obtained by approaching cuts from above:

$$A_s(s,t,u) = \lim_{\varepsilon \to 0} A(s+i\varepsilon,t,u)$$

$$A_u(s,t,u) = \lim_{\varepsilon \to 0} A(u+i\varepsilon,t,u)$$

$$A_t(s,t,u) = \lim_{\varepsilon \to 0} A(t+i\varepsilon,t,u)$$

iii) Due to unitarity relations, Invariant amplitudes are real below thresholds in all three channels.

Invariant amplitudes are real analytic functions with discontinuities across the cuts :

$$A_s(s+i\varepsilon,t,u)-A_s(s-i\varepsilon,\ u)=2i\cdot Im\ A(s+i\varepsilon,t,u)$$

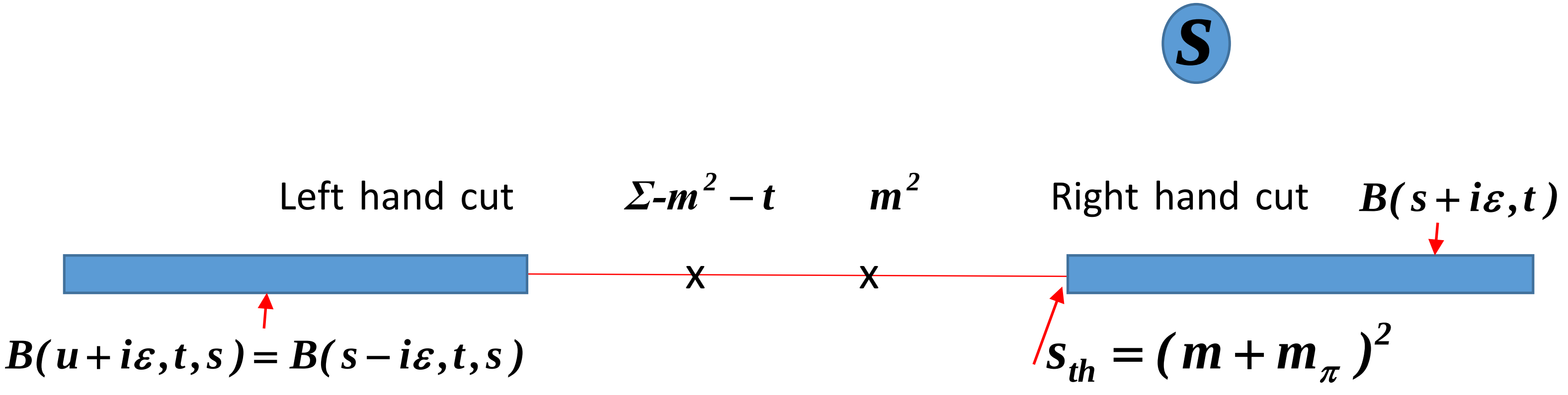
$$A_u(u+i\varepsilon,t,s)-A_u(u-i\varepsilon,\ t,\ u)=2i\cdot Im\ A(u+i\varepsilon,t,s)$$

$$A_t(t+i\varepsilon,s,u)-A_t(t-i\varepsilon,\ s,\ u)=2i\cdot Im\ A(t+i\varepsilon,s,u)$$

In practical applications one of the variables is kept constant so that the amplitudes are functions of only one variable.

Important case is $t=const$

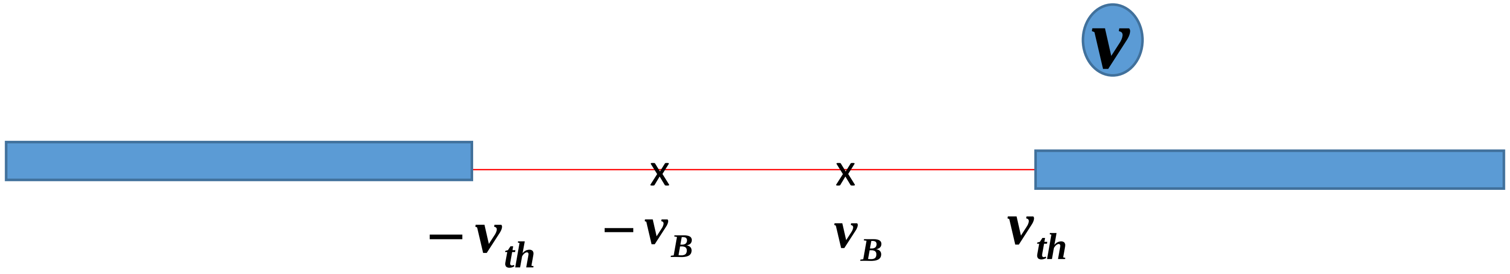
## Singularities of B amplitude for fixed t- variable.




It is more practical tu use so called crossing variable variable $\nu = \frac{s-u}{4\cdot m}$

S-u crossing implies sign change $\nu \to -\nu$ .

In a complex $\nu$ plane aplitude *B has analytical structure shown in dawing.*



In applications it is convenient to work with so called isospin even and odd amplitudes:

$$A^{\pm} = \frac{1}{2}(A_{-} \pm A_{+}), \quad B^{\pm} = \frac{1}{2}(B_{-} \pm B_{+}),$$

Under s - u crossing these amplitudes behave as:

$$A_s^{\pm}(s,t,u) = \pm A_u^{\pm}(u,t,s), \quad B_s^{\pm}(s,t,u) = \mp B_u^{\pm}(u,t,s),$$

Invariant amplitude $C$ defined as:

$$C^{\pm}(v,t) = A^{\pm}(v,t) + \frac{v}{1 - \frac{t}{4m^2}} B^{\pm}(v,t)$$

# Application of fixed t analyticity of invariant amplitudes in PWA

- One of the main problems of SE PWA are ambiguities
  ie more sets of partial waves equally well describe experimental data.
- First attempt: Requiring smoothnes of IA as a function of energy.
  It was shown that it was not enough.

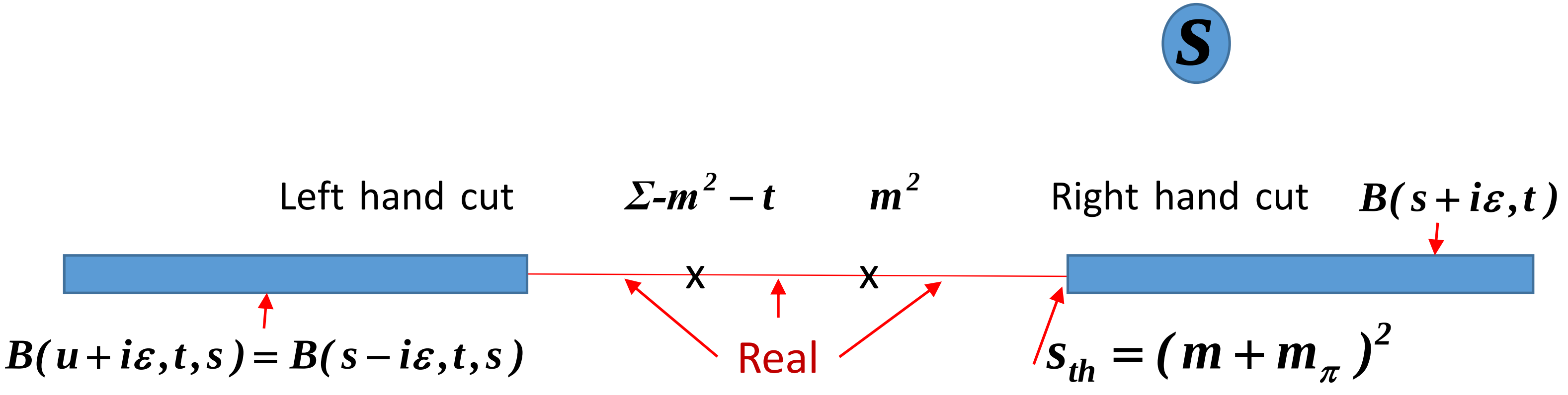
- One must impose more stringent constraints taking
  into account analayticity of scattering amplitudes.
- In this lecture:
  We demonstrate how the fixed-t analyticity
  can be used for that purpose.

We shall apply two methods:

- Fixed-t dispersion relations
- Analytic approximations theory

J. E. Bowcock and H. Burkhardt
Rep. Prog. Phys. 38 (1975)1099

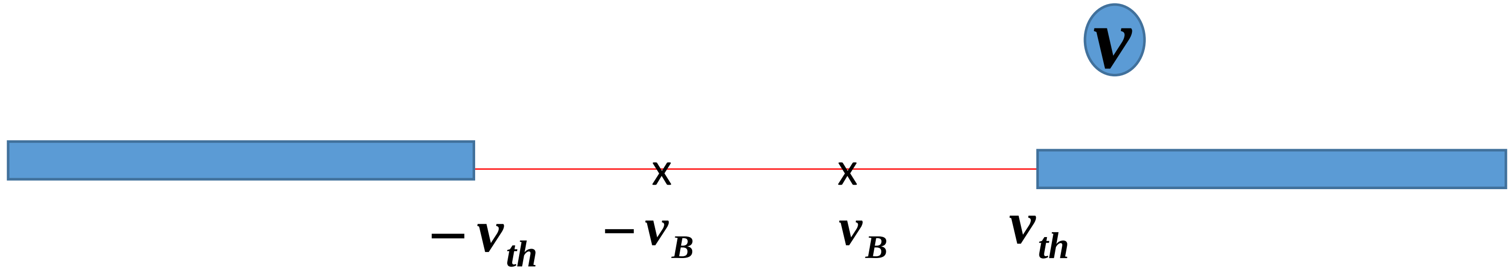
**Singularities of B amplitude for fixed t- variable.**




It is more practical to use so called crossing variable variable $\nu = \dfrac{s-u}{4\cdot m}$

s-u crossing implies sign change $\nu \rightarrow -\nu$ .

In a complex $\nu$ plane aplitude ***B** has analytical structure shown in a drawing.*




$$\nu_{th} = m_\pi + \frac{t}{4m}$$

$$\nu_B = \frac{(t-2m_\pi)^2}{4m}$$

In applications it is convenient to work with so called isospin even and isospin odd amplitudes :

$$A^{\pm} = \frac{1}{2}(A_{-} \pm A_{+}), \quad B^{\pm} = \frac{1}{2}(B_{-} \pm B_{+}).$$

$$A^{+} = \frac{1}{3}\left(A^{1/2} + 2A^{3/2}\right), \quad A^{-} = \frac{1}{3}\left(A^{1/2} - A^{3/2}\right)$$

Under ***s-u*** crossing these amplitudes behave as :

$$A_s^{\pm}(s,t,u) = \pm A_u^{\pm}(u,t,s), \quad B_s^{\pm}(s,t,u) = \mp B_u^{\pm}(u,t,s),$$

In most cases invariant amplitudes $C^{\pm}$ defined as :

$$C^{\pm}(v,t) = A^{\pm}(v,t) + \frac{v}{1 - \frac{t}{4m^2}} B^{\pm}(v,t)$$

are used instead of amplitudes $A^{\pm}$.

# Partial wave expansion of $\pi N$ invariant amplitudes

In numerical calculations one starts with amplitudes $f_1$ and $f_2$ which are simply related to partial wave amplitudes (partial waves) :

$$f_1(s,\cos\theta) = \frac{1}{q}\sum_{l=1}^{\infty}\left\{T_{(l-1)+}(s) - T_{(l+1)-}(s)\right\}P_l^{'}(\cos\theta)$$

$$f_2(s,\cos\theta) = \frac{1}{q}\sum_{l=1}^{\infty}\left\{T_{l-}(s) - T_{l+}(s)\right\}P_l^{'}(\cos\theta)$$

Invariant amplitudes $A$ and $B$ are related to amplitudes $f_1$ and $f_2$ in the following way :

$$\frac{A}{4\pi} = \frac{W+m}{E+m}f_1 - \frac{W-m}{E-m}f_2$$

$$\frac{B}{4\pi} = \frac{f_1}{E+m} - \frac{f_2}{E-m}$$

$E$ = Energy of proton in CMS

$W$ = Total energy in CMS

$m$ = nucleon mass

## Application of the fixed-t analyticity of Invariant amplitudes to partial wave analysis (PWA)

- **One of the main problems of single energy PWA (SE PWA) are ambiguities i.e. more sets of partial waves equally well describe experimental data.**
- **First attempt: Requiring smoothnes of IA or partial waves as a function of energy. It was shown that it was not enough.**

- **One must impose more stringent constraints taking into account analyticity of scattering amplitudes.**
- **In this lecture:** **We demonstrate how the fixed-t analyticity can be used for that purpose.**

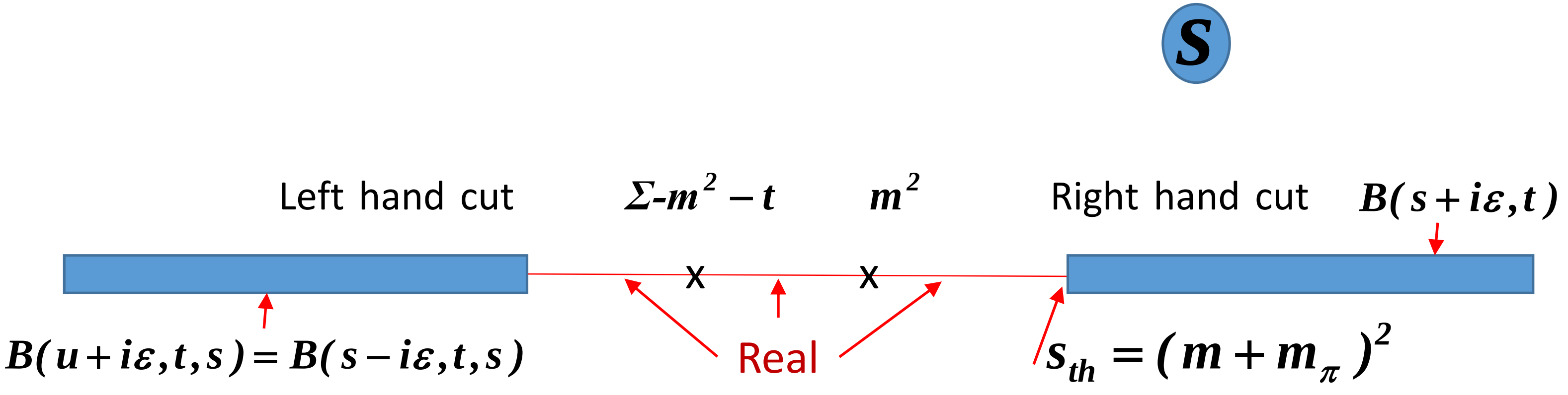
**We shall apply two methods:**

- **Fixed-t dispersion relations (DR)**
- **Analytic approximation theory**

J. E. Bowcock and H. Burkhardt Rep. Prog. Phys. 38 (1975)1099

# Fixed-t dispersion relations

**Singularities of B amplitude for fixed t- variable.**

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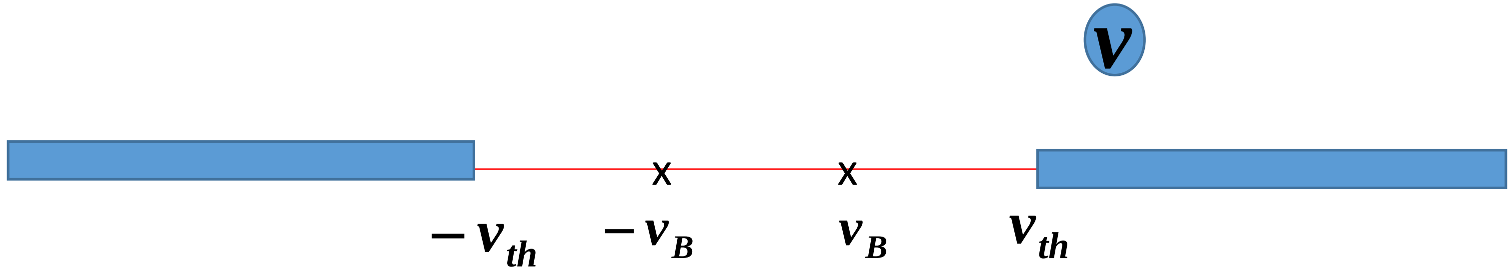
In a complex $\nu$ plane aplitude $B$ *has analytical structure shown in a drawing.*




$$\nu_{th} = m_\pi + \frac{t}{4m}$$

$$\nu_B = \frac{(t-2m_\pi)^2}{4m}$$

# Fixed-t dispersion relations

Let's consider DR fulfilled by a typical isospin even ( odd ) $\pi N$ invariant amplitude (IA) :

$$Re\,F^{\pm}(\nu,t) = F_N^{\pm}(\nu,t) + \frac{1}{\pi}\int_{\nu_{th}}^{\infty} d\nu'\, Im\,F^{\pm}(\nu',t)\left(\frac{1}{\nu'-\nu} \pm \frac{1}{\nu'+\nu}\right)$$

$F_N$ - contribution from nucleon pole

- Suppose that results from SE PWA are available in the range $\nu_{th} < \nu < \nu_h$ and smoothly interpolated ( using spline interpolation for instance).
- Dispersion relations may be written in the form:

$$Re\,F^{\pm}(\nu,t) = F_N(\nu,t) + \frac{1}{\pi}\int_{\nu_{th}}^{\nu_h} d\nu'\, Im\,F^{\pm}(\nu',t)\left(\frac{1}{\nu'-\nu} \pm \frac{1}{\nu'+\nu}\right)$$

$$+ \frac{1}{\pi}\int_{\nu_h}^{\infty} d\nu'\, Im\,F^{\pm}(\nu',t)\left(\frac{1}{\nu'-\nu} \pm \frac{1}{\nu'+\nu}\right)$$

$$Re\,F^{\pm}(\nu,t) = F_N(\nu,t) + \frac{1}{\pi}\int_{\nu_{th}}^{\nu_h} d\nu'\, Im\,F^{\pm}(\nu',t)\left(\frac{1}{\nu'-\nu} \pm \frac{1}{\nu'+\nu}\right) + \Delta_F(\nu,t) \qquad (1)$$

$\Delta_F(\nu,t)$ describes unknown contributions from high energies and is called discrepancy function (DF).

H. Nielsen, Nucl. Phys., B30 (1971) 317
B33 (1971) 152
H. Nielsen, G.C. Oades,
Nucl. Phys., B49 (1972) 573
B49 (1972) 586

J. Hamilton, J. L. Petersen in
New developments in dispersion theory Vol. 1,
Nordita, Copenhagen 1975

- Solving last equation for $\Delta_F$ ( dropping variable *t from the list of variables*), and using known values of

  $Re\,F^{\pm}(v_k),F_N^{\pm}(v_k)$ and $PVI(v_k)$

  we obtain:

  $$\Delta_F(v_k)=Re\,F^{\pm}(v_k)-F_N^{\pm}(v_k)-PVI(v_k),$$

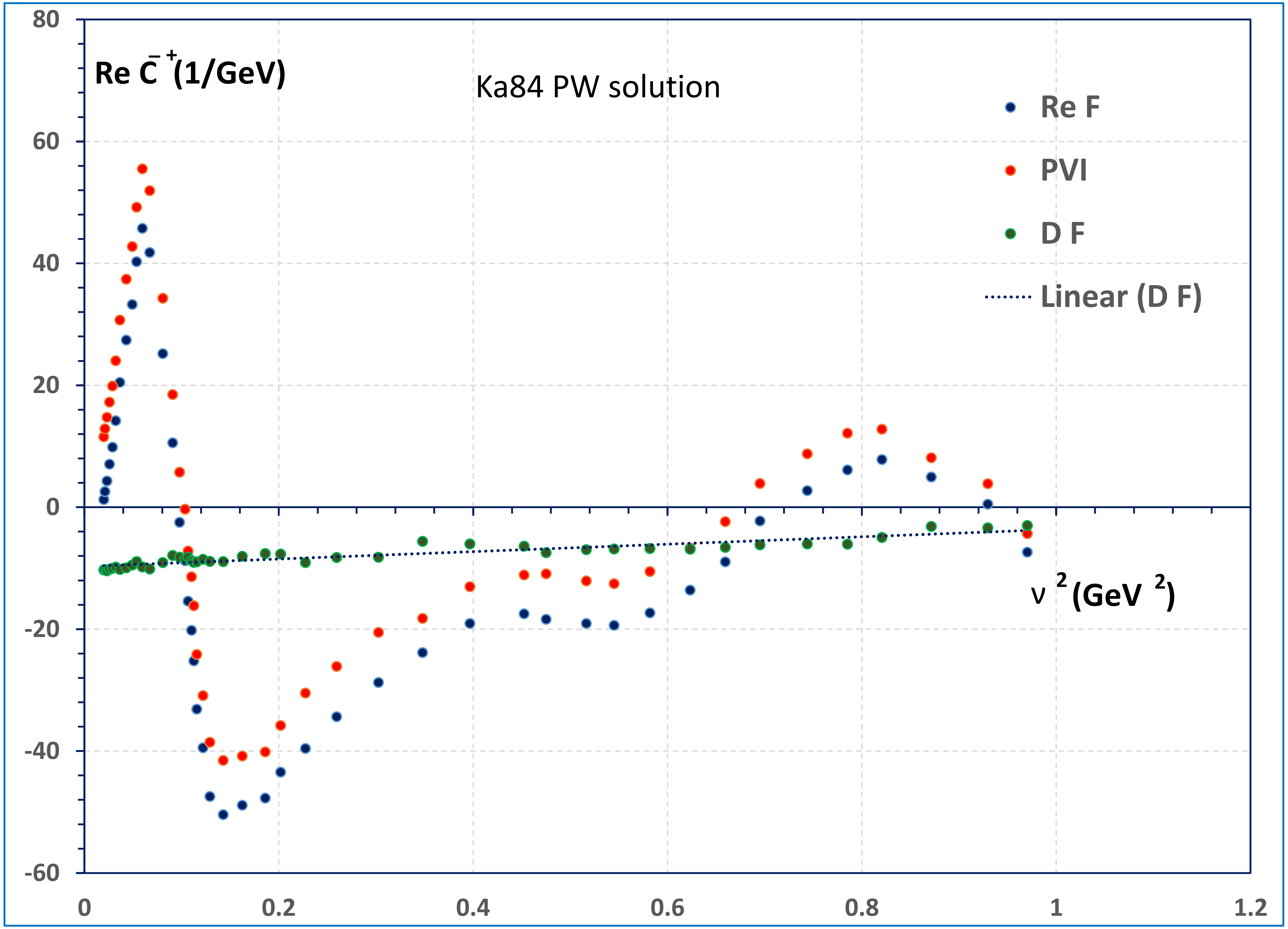
  where PVI stands for principal value integral.

- $\Delta_F$ may be approximated by a polynomial of maximal order two !

- Accurate determination of the high energy contributions is a big advantage of this method.

- Using parametrized $\Delta_F$ in DR (*) one obtains a smooth real part of the amplitude $Re\,F$

- Discribed method, called Discrepancy function method, may be used to make accurate analytic continuation of IA outside of the physical region.



Re C̄⁺(1/GeV)
Ka84 PW solution
Re F
PVI
D F
Linear (D F)
ν ²(GeV ²)
80
60
40
20
0
-20
-40
-60
0
0.2
0.4
0.6
0.8
1
1.2

## How tu use DF method as a constraint in PWA?

1. Suppose results of SE PWA are available in energy region $w_{th} < w < w_h$ corresponding to $v_{th} < v < v_h$
   Evaluate $Re\,\hat{F}_k\,(k = 1{,}4$ stand for four IA $)$ in small steps of $t$ (say $n$)
   over a range $0 \le t \le -2q^2\ (-1 \le cos\,\theta \le 1)$, where $q$ is a pion momentum in CMS.
2. In PWA program minimize:

   $$\chi^2 = \chi^2_{data} + \chi^2_{FT}$$

   where $\chi^2_{data}$ for an observable $D$ measured at $N_D$ angles :

   $$\chi^2_{data} = \sum_{k=1}^{N_D} \left( \frac{D_{exp}(\theta_k) - D_{PW}(\theta_k)}{\Delta_k} \right)^2 ,$$

   where $D_{PW}$ is a value of measurable quantity $D$ calculated from partial waves which are parameters in a fit.

   A term $\chi^2_{FT}$ has a form :

   $$\chi^2_{FT} = \sum_{k=1}^{4} \sum_{j=1}^{n} \left( \frac{(Re\,\hat{F}_k(t_j) - F_{k\,PW}(t_j))}{\varepsilon_{kj}} \right)^2$$

3. After finishing PWA part, repeat step 1. Carry out the iteration until achieving a reasonable agreement between input and output values .

In DF method, one may constrain only a real part of the amplitude. One expects that the data will make a right choice of imaginary parts of partial waves.

In elastic region of πN scattering DF method was applied as a strong constraint because real and imaginary parts can be derived from eah other due to unitarity of partial waves.

Problems with applying DR as a constraint:

- Errors of PW are generally strongly correlated and are usually not published.
- It is not clear how to calculate errors of PVI.
  Small changes in the input part can lead to a large change in the output real part.

But:

- DR can serve as an simple test of compatibility of PW with the fixed- t analyticity.
  Too noisy PW (non smooth DF- higher order polynomials needed to describe DF) are indicator that SE PWA solution is not consistent with FT analyticity of scattering amplitudes.

# Analytic approximation theory

Fixet-t analyticity of scattering amplitudes can be imposed without using of DR. Invariant amplitudes can be obtained from the data directly.

Consider an amplitude $F(v)$ which has the following properties :

*i)* $F(v)$ is real analytic function in the complex $v$ plane cut along $(v_{th}, +\infty)$

*ii )* $F(v)$ is bounded through $v$ plane

Let's start with the simplest case when experimental information on the real and imaginary parts are available at a finite number of points.

*i)* Assume that real parts of $F(v)$ has been 'measured' at M points $v_1,...,v_m$ giving values $f_1,..., f_M$ and corresponding errors $\varepsilon_1,...,\varepsilon_M$.

*ii )* Assume that the imaginary part $F(v)$ has been measured at the N - M points $v_{M+1},...,v_N$ giving values $f_{M+1},..., f_N$ and corresponding errors $\varepsilon_{M+1},...,\varepsilon_N$.

In ***most*** problems we have data on $|F(v)|^2$ or some other bilinear combinations.

**This part of lecture is based on:**

J. Hamilton, J. L. Petersen in New developments in dispersion theory Vol. 1, Nordita, Copenhagen 1975

E. Pietarinen, Nuovo. Cim. 12 (1972) 522
Nucl. Phys. B49 (1972) 315
Nucl. Phys B107(1976) 21,
end references there

## Finding optimal approximant

Let $\varphi(v)$ is an analytic function with the analytic properties similar to $F(v)$.

Our task is following :

Find optimal approximant $\varphi$ for the given data on $F$.

We proceed in a standard way ( The method of least squares ):

$$\chi^2(\varphi) = \sum_{k=1}^{M} \left( \frac{Re\,\varphi(v_k) - f_k}{\varepsilon_k} \right)^2 + \sum_{k=M+1}^{N} \left( \frac{Im\,\varphi(v_k) - f_k}{\varepsilon_k} \right)^2$$

There are infinitely many functions giving $\chi^2 = 0$.

In general, they will be ***extremely unsmooth*** outside the data region.

Such functions are unacceptable candidates for a physical amplitude.

We want approximant to be smooth and look for optimal approximant which is the 'smoothest' function that has an acceptable $\chi^2$.

We define a measure $\Phi(\varphi)$ of 'lack of smoothness' of the approximant $\varphi$.

$\Phi(\varphi)$ is called the ***penalty function*** or the ***convergence test function***.

The optimal approximant is defined to be one that minimizes :

$$X^2 = \chi^2(\varphi) + \Phi(\varphi)$$

## Digression - Conformal mapping method

Suppose we have a complex function which is analytic in $z$ plane except the cuts shown in the figure.

Conformal mapping :

$$w = \frac{1 - \sqrt{\frac{b\,(a-z)}{a\,(b+z)}}}{1 + \sqrt{\frac{b\,(a-z)}{a\,(b+z)}}}$$

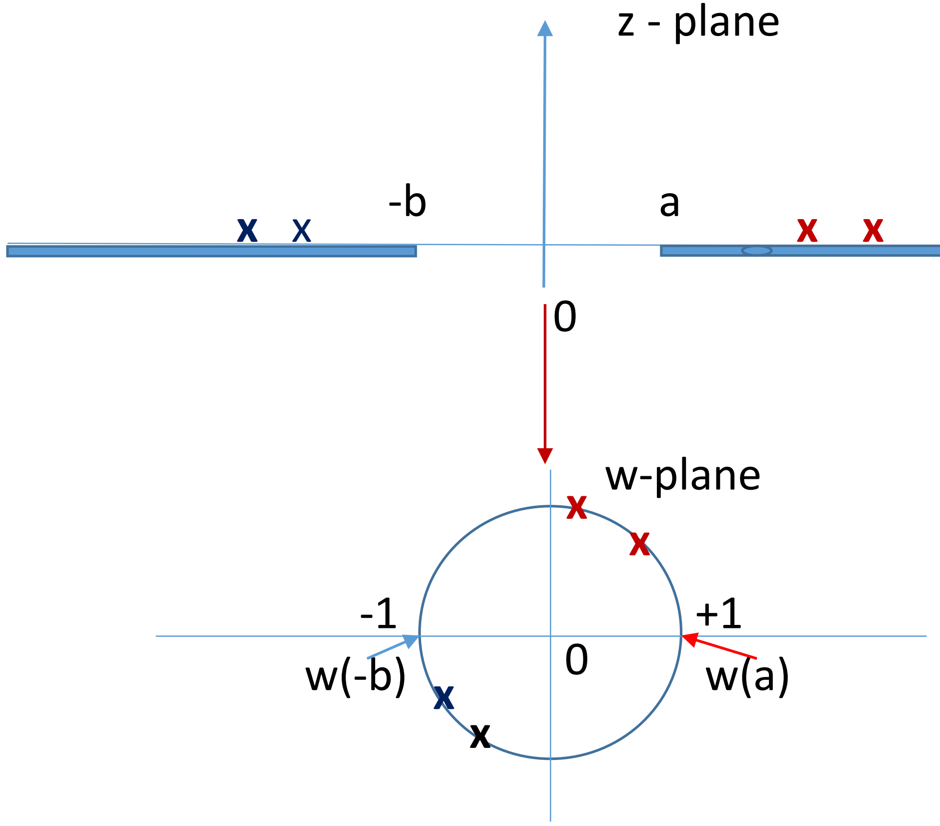
maps the cut $z$ - plane into and on the unit circle as shown in the figure.

One can expand function ***f(z)*** in a power series in $w$ about the origin in $w$ plane :

$$f(z) = \sum_{n=0}^{\infty} c_n w^n .$$

If a function is bounded, then power series converge on the unit circle. It means that expansion represents function ***f(z)*** in a whole cut z - plane.

The power series in $w$ ***generally converge more rapidly*** than the ordinary power series in $z$ .




W. R. Fraser, Phys. Rev 123 (1961) 2180
See also example on page 567 in:
C. B. Lang, N. Pucker, Mathematische Methoden in der Physik, 2nd Ed., Springer Spektrum, Berlin 2005.

Conformal mapping :

$$z = \frac{\alpha - \sqrt{v_{th} - v}}{\alpha + \sqrt{v_{th} - v}}, \qquad \alpha \text{ real} > 0$$

maps the cut $v$ plane to interior and onto the unit circle.

Any function bounded in $v$ plane may be represented by

Taylor series converging inside and on the unit circle $|z| = 1$ :

$$\varphi(v) = \varphi(z) = \sum_{n=0}^{\infty} a_n z^n .$$

Using arguments from probability theory, Pietarinen has proposed a penalty function in the form :

$$\Phi(\varphi) = \lambda \cdot \sum_{n=0}^{\infty} (n+1)^3 a_n^2, \qquad (*)$$

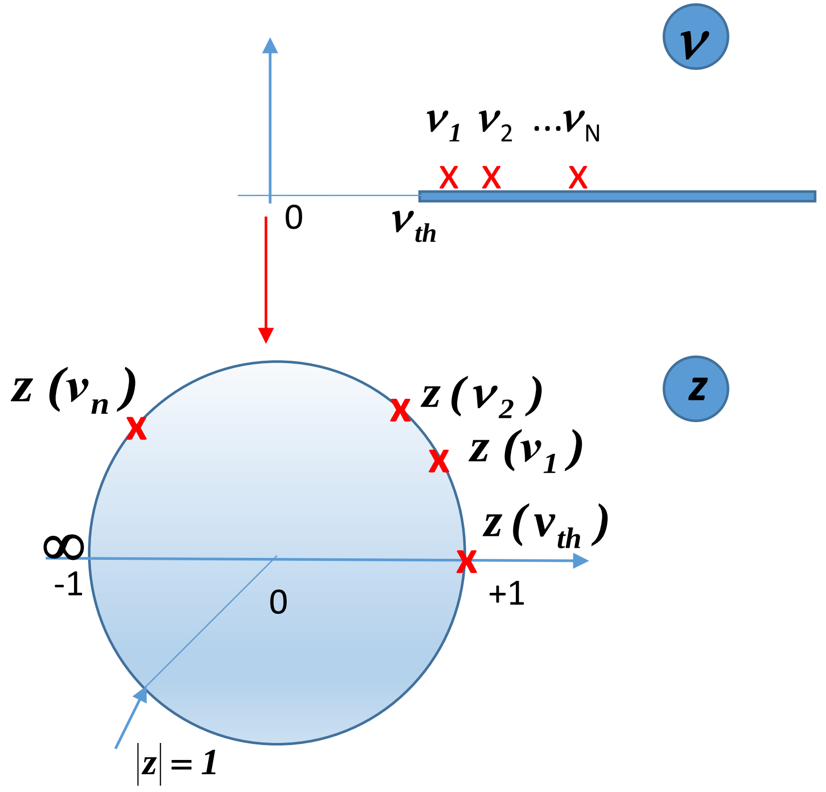
where $\lambda$ is a real scaling parameter.

It can be shown that coefficients $a_n$ are strongly suppressed :

$$|a_n| \le \frac{1}{n^{\frac{3}{2}} \cdot \lambda},$$

Expansion $(*)$ may be truncated at the finite order $N_{max}$.

Minimizing $X^2$ is a ***compromise*** between fitting the data and keeping coefficients in $\Phi(\varphi)$ small.



Minimizing $X^2$

Minimizing $X^2$

Using expansion for $\varphi(v)$ and a formula for penalty function $\Phi$, $X^2(\varphi)$ may be written in the form:

$$X^2(\varphi) = \sum_{l=0}^{M}\sum_{m=1}^{M} A_{lm} a_l a_m - 2\sum_{l=0}^{M} B_l a_l + \sum_{k=1}^{N} \frac{f_k^2}{\varepsilon_k^2} + \lambda \sum_{l=0}^{N_{max}} (l+1)^3 a_l^2$$

where:

$$A_{lm} = \sum_{k=1}^{M} \frac{Re(z_k^l)\cdot Re(z_k^m)}{\varepsilon_k^2} + \sum_{k=m+1}^{N} \frac{Im(z_k^l)\cdot Im(z_k^m)}{\varepsilon_k^2}$$

$$B_l = \sum_{k=1}^{M} \frac{f_k\, Re(z_k^l)}{\varepsilon_k^2} + \sum_{k=1}^{M} \frac{f_k\, Im(z_k^l)}{\varepsilon_k^2}$$

$$\frac{\partial X^2(\varphi)}{\partial a_l} = 2\cdot\sum_{k=1}^{M} A_{lm} a_m + 2B_l + \lambda\cdot\sum_{l=0}^{N_{max}} (l+1) a_l = 0 \quad l = 0,\ldots,N_{max}$$

Coefficients of the best approximant are solution of a set of linear equation written in a matrix form:

$$(A + \lambda D)\cdot \bar{a} = B$$

The error matrix $E$ is inverse of $(A+\lambda D)$:

$$E = (A + \lambda D)^{-1}.$$

Errors of coefficients $\bar{a}_n$:

$$\Delta \bar{a}_n = \sqrt{E_{nn}}$$

Determination of the scaling parameter $\lambda$

According to Pietarinen, scaling parameter is given by formula :

$$\lambda = \frac{N_{max}}{\sum_{n=0}^{N_{max}} (n+1)^2 (a_n^2 + E_{nn})}$$

Procedure :

*i)* Start with some reasonable value of $\lambda$ and determine coeff. $a_n$ and corresponding $E_{nn}$

*ii )* Calculate a new $\lambda$

*iii)* Iterate. Iteration converges in a few steps.

In practical analysis :

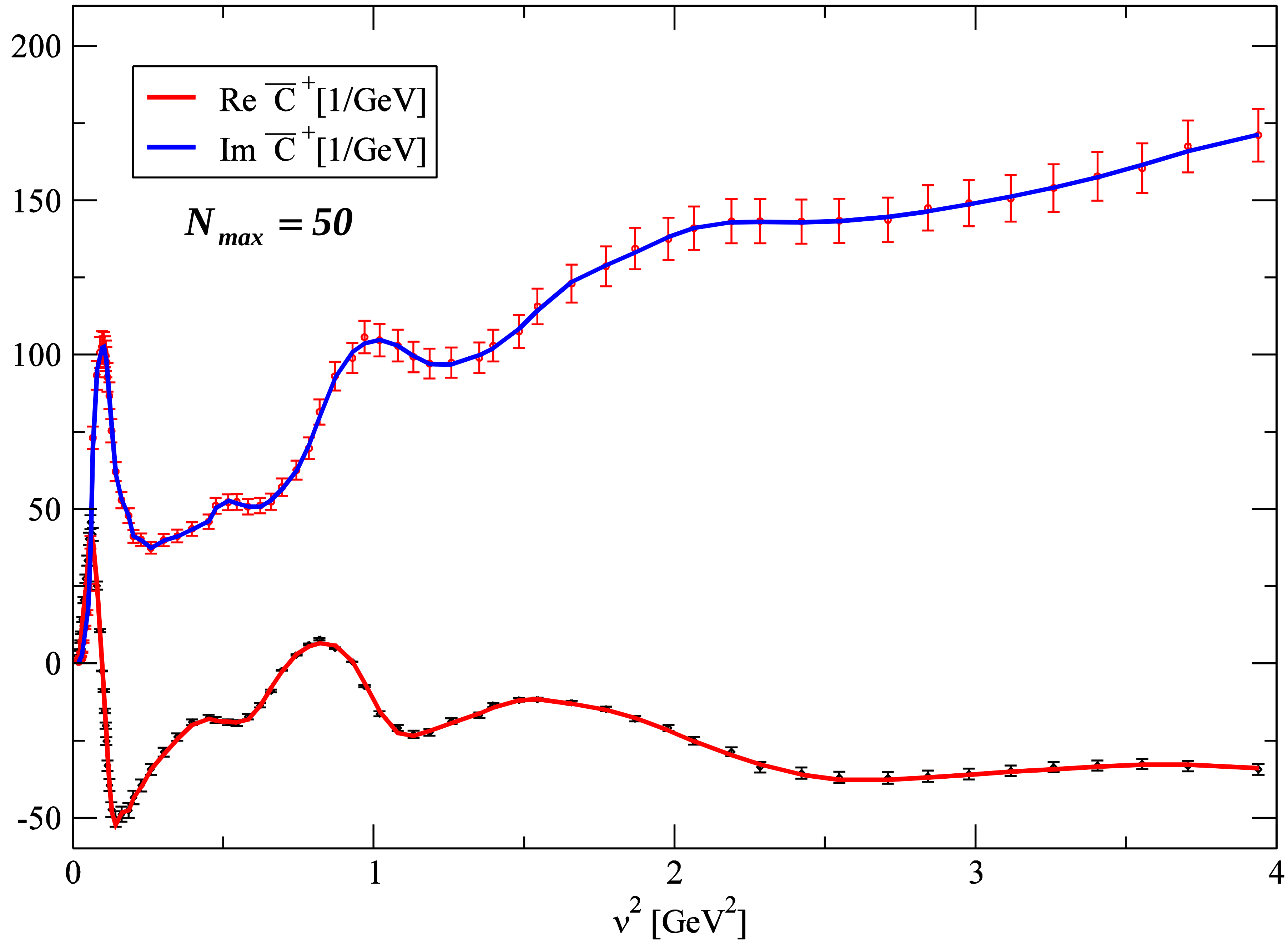
$N_{max} \approx 30$

$$\lambda \approx \frac{N_{max}}{\sum_{n=0}^{N_{max}} (n+1)^2 \cdot a_n^2}$$

In his Nuovo Cimento paper Pietarinen applied his method and tested it on the forward $\pi N$ scattering amplitudes $C^{\pm}$ due to their simple connection to the total cross section $Im\, C^{\pm} \sim \sigma_{tot}^{\pm}$

Application of Pietarinen's method to $\pi^+\pi^-$ scattering with detailed description of all steps in their analysis may by found in :

C. D. Froggatt and J. L. Petersen, Nucl. Phys. B91 (1975) 454



Re $\overline{C}^+$[1/GeV]
Im $\overline{C}^+$[1/GeV]
$N_{max} = 50$
200
150
100
50
0
-50
0
1
2
3
4
$\nu^2$ [GeV$^2$]

# Pietarinen's $\pi N$ fixed-$t$ amplitude analysis

Pietarinen's fixed-$t$ (FT) amplitude analysis is one of the main analyticity constraints in Karlsruhe-Helsinki $\pi N$ PWA.

In his analysis Pietarinen used IA $C^{\pm}$ and $B^{\pm}$.

At a fixed-$t$ IA amplitudes $C^{+}, B^{-}, C^{-}/v$ and $B^{+}/v$ are crossing symmetric-even function of $v$.

Appart from nucleon poles, crossing symmetric IA are analytic functions in a complex $v^2$ plane cut along $v_{th}^2 \leq v^2 < \infty$.

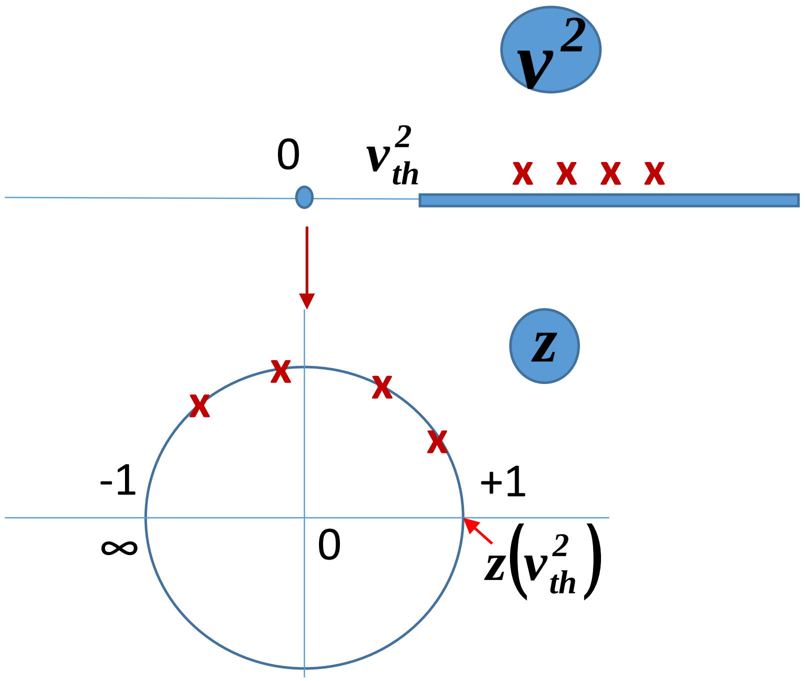
IA are not measured directly. They enter quadratically or in some bilinear form in expressions for measurable quantities.

Using conformal mapping:

$$z = \frac{\alpha + \sqrt{v_{th}^2 - v^2}}{\alpha - \sqrt{v_{th}^2 - v^2}}, \qquad \alpha > 0, \text{real}.$$

a cut $v^2$ plane is mapped inside and on the unit circle.

A suitable value of parameter $\alpha$ makes a more uniform distribution of data points on the unit circle.



Representation of invariant amplitudes

Representation of invariant amplitudes (IA)

$C^{\pm}$ and $B^{\pm}$ are represented in a following way:

$$C^{\pm}(v,t) = C^{\pm}_{N}(v,t) + \hat{C}^{\pm}(v,t) \cdot \sum_{n=0}^{N} c_n^{\pm} z^n$$

$$B^{\pm}(v,t) = B^{\pm}_{N}(v,t) + \hat{B}^{\pm}(v,t) \cdot \sum_{n=0}^{N} b_n^{\pm} z^n$$

E. Pietarinen,
Nucl. Phys. B107 (1976) 21

$C_N^{\pm}$ and $B_N^{\pm}$ are corresponding nucleon poles. Factors $\hat{C}^{\pm}$ and $\hat{B}^{\pm}$ describe high energy behaviour of corresponding amplitudes.

The best approximants of IA are determined by minimizing a quadratic form:

$$X^2 = \chi^2_{data} + \Phi,$$

where:

$$\Phi = \lambda_1 \Phi_1(c_n^+) + \lambda_2 \Phi_2(c_n^-) + \lambda_3 \Phi_3(b_n^+) + \lambda_4 \Phi_4(b_n^-)$$

Choice of scaling parameter $\lambda$

$$\lambda_1 = \frac{N}{\sum_{n=0}^{N} \left(c_n^+\right)^2 (n+1)^3},$$ similar for others, $N \approx 30$

Steps in the FT amplitude analysis

Steps in the fixed - t amplitude analysis

i) Prepare input for t = 0. This is most important part of the input because imaginary parts of $C^{\pm}$ IA are directly connected to the $\pi^{\pm}N$ total cross sections.

ii) Move experimental data to predefined t - values using small steps in t. Problem is not trivial - it might be a subject of another talk.

iii) Start with amplitude analysis for t = 0. Use obtained coefficients in amplitude expansions as starting values in analysis for next t - value.

iV) Continue until reaching minimum t - value. Pietarinen made his FT amplitude analysis at 40 t - values in the range $-1.0\ GeV^2 \le t \le 0.$

V) Results from FT amplitude analysis consists of coefficients $\left\{c_n^{\pm}, b_n^{\pm}\right\}$ for all t - values used in the analysis.

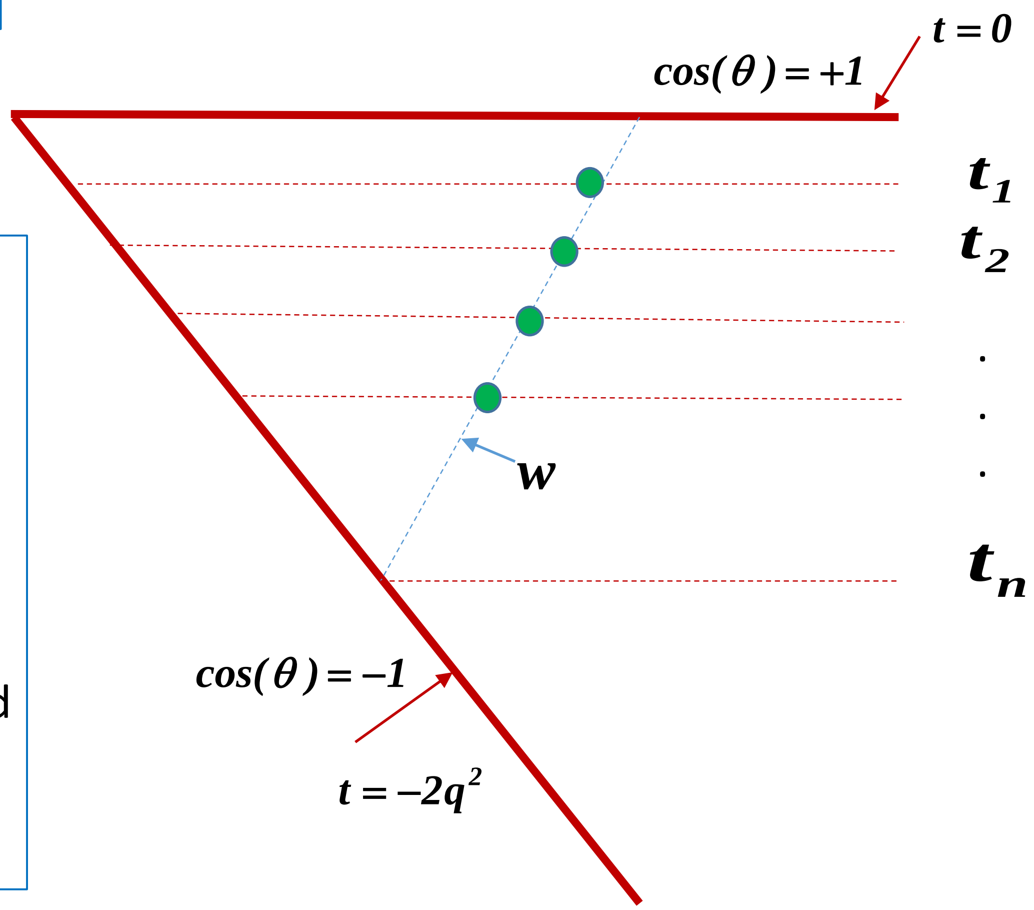
Calculation of IA is reliable and fast and is performed by nested multiplication.

Problem : Errors obtained from fitting programs are correlated and underestimated - another problem for itself.

## SE PWA constrained by results from fixed-*t* amplitude analysis

Available experimental data have to be moved to predetermined energy values - fixed energy data bins. Suppose that we perform SE PWA at energy w.

Let the range $-2q^2 \le t \le 0$ $(-1 \le \cos\theta \le +1)$ contains $n$ $t$-values at which FT analysis was performed. Obtained results are denoted by $\hat{F}_k(t_j)(k = 1,4,\ j = 1,\ldots n)$.



## Constrained SE PWA

i) In SE PWA minimize :

$$\chi^2 = \chi^2_{data} + \chi^2_{FT}$$

For an observable $D$ measured at $N_D$ angles $\chi^2_{data}$ has a form

$$\chi^2_{data} = \sum_{k=1}^{N_D} \left( \frac{D_{exp}(\theta_k) - D_{PW}(\theta_k)}{\Delta_k} \right)^2$$

$D_{exp}(\theta_k)$ are measured data with errors $\Delta_k$.

$D_{PW}$ is a value of measurable quantity calculated from partial waves which are parameters in a fit.

$\chi^2_{FT}$ is constraining part in the form :

$$\chi^2_{\mathrm{FT}} = \sum_{k=1}^{4} \sum_{j=1}^{n} \left( \frac{(Re\,\hat{F}_k(t_j) - Re\,F_{k\,PW}(t_j)}{\varepsilon_{k\,j}} \right)^2 + \sum_{k=1}^{4} \sum_{j=1}^{n} \left( \frac{(Im\,\hat{F}_k(t_j) - Im\,F_{k\,PW}(t_j)}{\varepsilon_{k\,j}} \right)^2$$

Errors of FT amplitudes are not fixed.

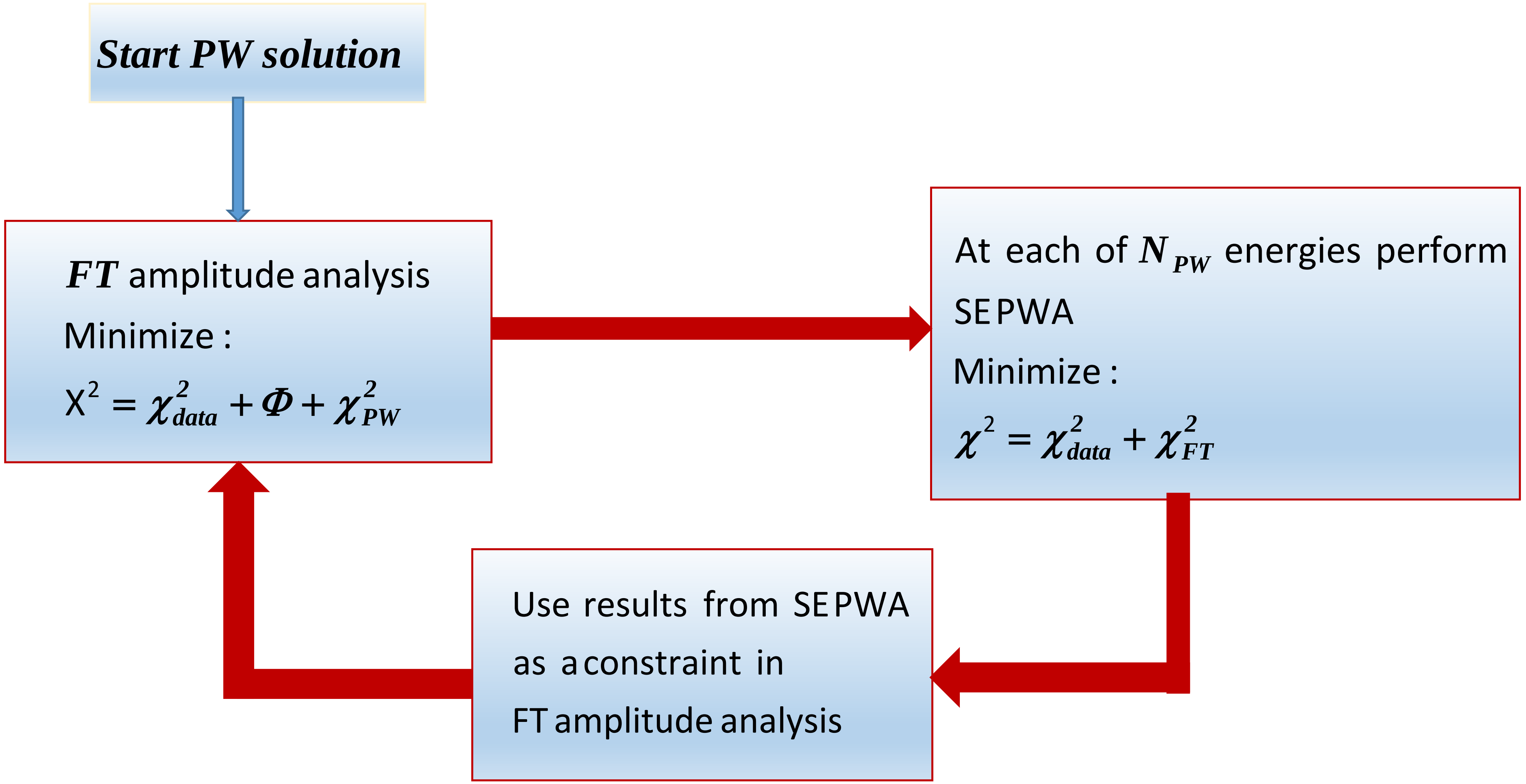
Should be adjusted so that in the final fit $\chi^2_{data} \approx \chi^2_{FT}$

ii) After performing SE PWA at predetermined $N_{PW}$ energies, use obtained partial waves as a constraint in a FT amplitude analysis minimizing :

$$X^2 = \chi^2_{data} + \Phi + \chi^2_{PW}$$

$$\chi^2_{PW} = \sum_{k=1}^{N_{PW}} \left( \frac{Re\,F(\nu_k) - Re\,F_{PW}(\nu_k)}{\varepsilon_k} \right)^2 + \sum_{k=1}^{N_{PW}} \left( \frac{Im\,F(\nu_k) - Im\,F_{PW}(\nu_k)}{\varepsilon_k} \right)^2$$

iii) Iterate until reaching reasonable agreement



Start PW solution
FT amplitude analysis
Minimize :
$X^2 = \chi^2_{data} + \Phi + \chi^2_{PW}$
At each of $N_{PW}$ energies perform SEPWA
Minimize :
$\chi^2 = \chi^2_{data} + \chi^2_{FT}$
Use results from SEPWA as a constraint in FT amplitude analysis