# Selected Topics in Complex Analysis as Tools in PWA

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Literature

S. Hassani, Mathematical Physics, Springer Verlag, 1999.

J. H. Mathews, W. Howell, Complex Analysis for Mathematics
and Eengineering, Jones& Bartlet Learning, 2010

H. A. Pristley, Introduction to Complex Analysis, 2nd ed., Oxford Univ. Press, 2003.

J. Hamilton, J. L. Petersen, New developments in dispersion theory, Vol.1, Nordita Copenhagen,1975

**This is not a Course in Complex Analysis**

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## How the lectures will be organized?

Part 1 **Complex analysis- repetitorium**

From definition of complex number
to contour integrals and analytic continuation

Part 2 **Analyticity of invariant scattering amplitudes as an constraint in PWA.**

Fixed-t DR- method of discrepancy function
Pietarinen's method of convergence test function

# Complex algebra

Def. Complex number is defined by an ordered pair of real numbers $x, y \in R$, $z = (x, y) = x + i \cdot y$ where $i = (0,1)$ is imaginary unit.

Complex conjugate number ***z**** is defined as:

$$z^* = (x, -y) = x - i \cdot y$$

Algebraic operations with complex numbers.

i) Addition

$$z = z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$= (x_1 + x_2) + i \cdot (y_1 + y_2)$$

$$z_1 + z_2 = z_2 + z_1$$

$$(z_1 + z_2) + z_3 = (z_1 + z_2) + z_3$$

$$z_1 = -z = (-x, -y)$$

$$z + (-z) = 0 = (0,0)$$

ii) Multiplication

$$z_1 \cdot z_2 = (x_1, y_1) \cdot (x_2, y_2)$$
$$= (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$$

$$z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$$

$$z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$$

iii) Division

$$\frac{z_1}{z_2} = \frac{x_1 + i \cdot y_1}{x_2 + i \cdot y_2} = \frac{x_1 + i \cdot y_1}{x_2 + i \cdot y_2} \cdot \frac{x_2 - i \cdot y_2}{x_2 - i \cdot y_2}$$

$$= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$

## Norm or modulus of complex number

**Def.** $$|z| = \sqrt{x^2 + y^2} = \sqrt{z \cdot z^*}, \qquad |z| \in R$$

For complex numbers $z_1, z_2 \in C$ hold following inequalities:

i) $|z_1 + z_2| \le |z_1| + |z_2|$ triangle inequality

ii) $|z|^2 \ge 0, \qquad |z| = 0 \Rightarrow z = 0$

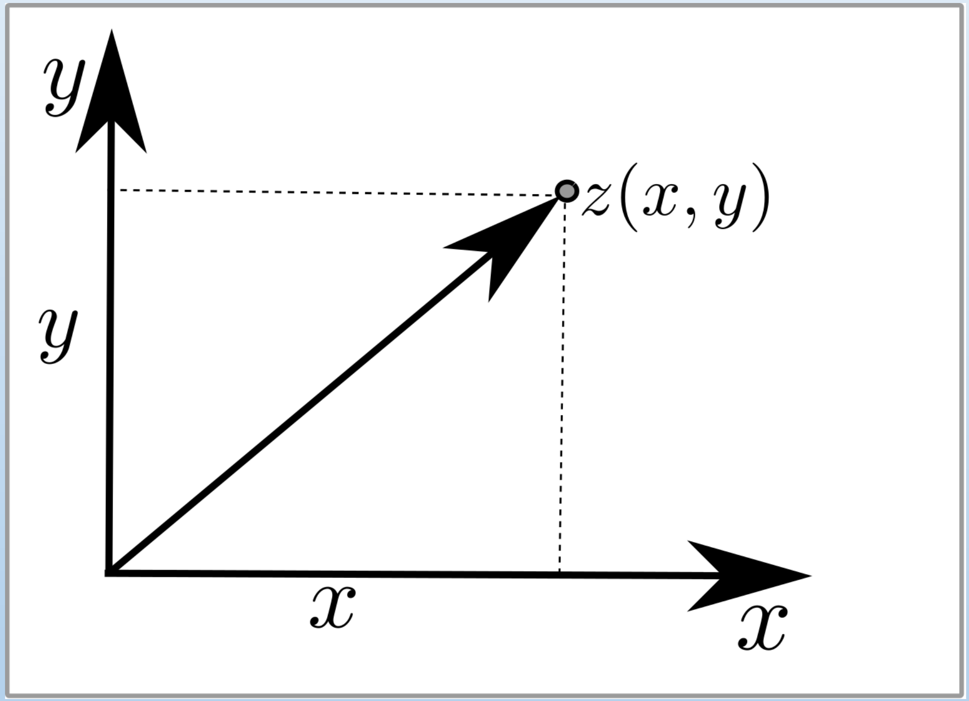
Additional useful inequalities:

iii) $|Re\, z| \le |z|; \qquad |Im\, z| \le |z|$

iV) $|z_1 + z| \ge \left||z_1| - |z_2|\right|$

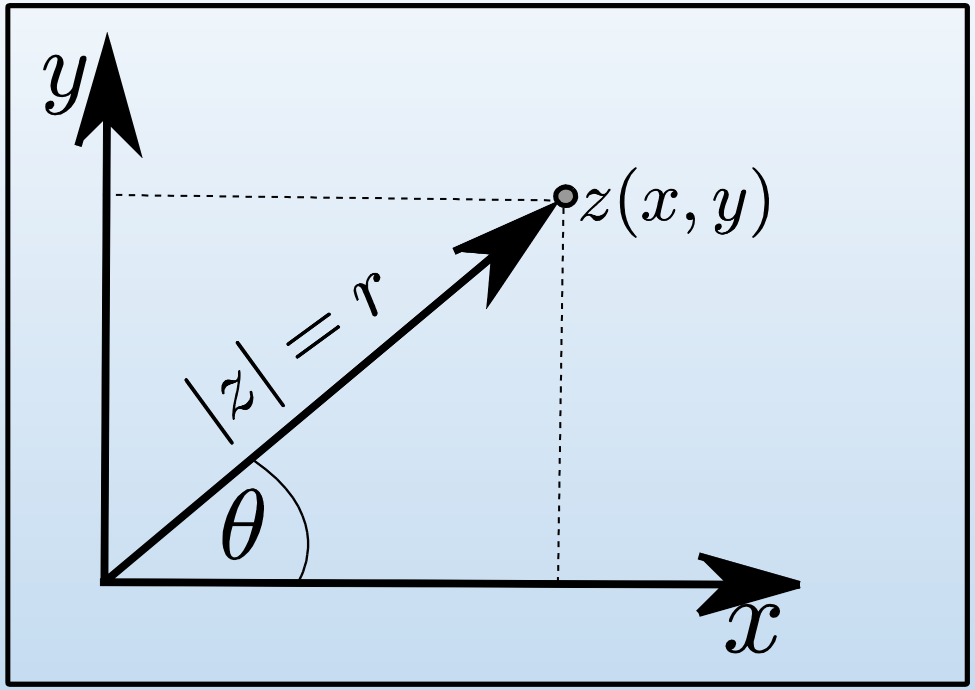
# Graphycal representation of complex numbers

Since a complex number ***z=(x,y)*** is an ordered pair of real numbers, It may be represented by point in *x,y* plane called complex plane.




***x*** and **y** axes are called real and imaginary axes and complex plane – z plane

## Polar representation of complex number




***Arg z is*** determined up to integer multiple of $2\pi$
Principal range of argument:

$$-\pi < \theta \leq \pi$$

$$x = r \cdot \cos\theta$$

$$y = r \cdot \sin\theta$$

$$z = r(\cos\theta + i \cdot \sin\theta) = r \cdot e^{i\theta}$$

$$\theta = \arg z$$

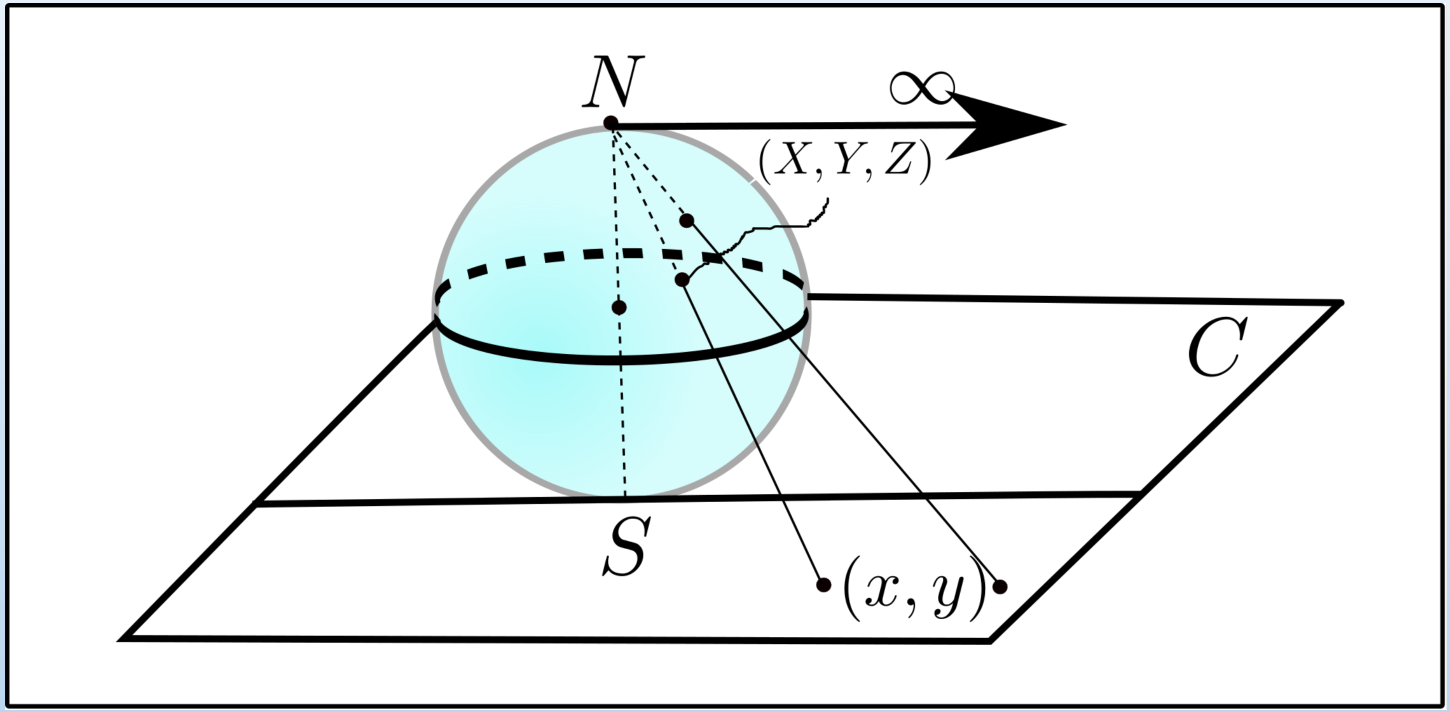
$$e^{2\pi \cdot i} = 1$$

Using complex numbers $z_1, z_2$ in polar form, multiplication and division are written in simple form:

$$z_1 \cdot z_2 = r_1 \cdot r_2 \cdot e^{i(\theta_1 + \theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot e^{i(\theta_1 - \theta_2)}$$

## Spherical representation of complex number
## Extended complex plane




Unique point N(0,0,1) corresponds to a point in infinity.
South pole coresponds to z=0.

The set of complex numbers including Point at infinity is called ***extended complex plane***

To each complex number *z=x+iy in complex plane C corresponds unique point on a unit sphere:*

$$X = \frac{x}{1 + x^2 + y^2}, \quad Y = \frac{y}{1 + x^2 + y^2}, \quad Z = \frac{x^2 + y^2}{1 + x^2 + y^2}$$

# Subsets in a complex plane

i) Real axis: $Im\, z = 0; \quad z = z^*$

ii) Imaginary axis: $Re\, z = 0$

iii) Line segment with end points $z_1, z_2 \in C$

$$z(t) = (1-t)\cdot z_1 + t\cdot z_2, \qquad 0 \le t \le 1$$

iv) Circle of radius $r$ with center in $z_0$

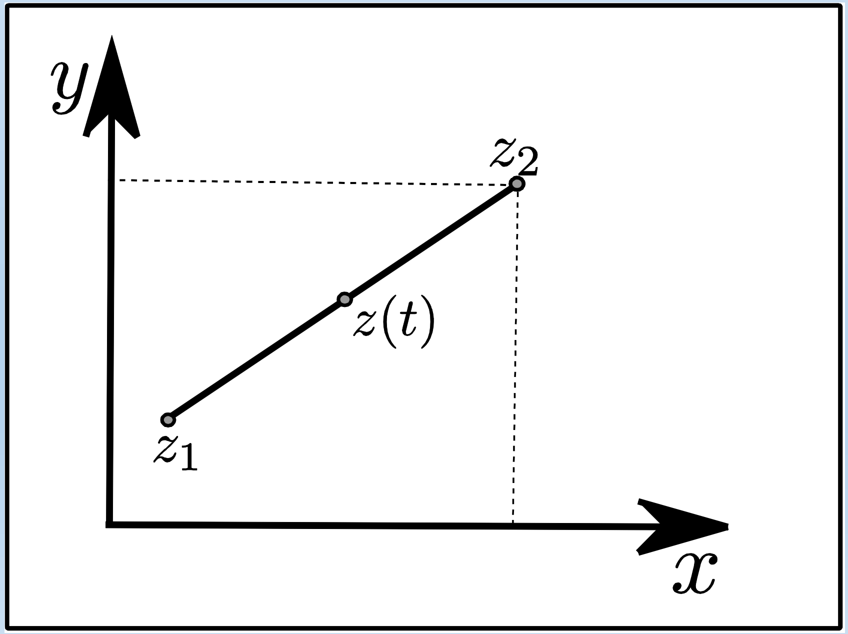
$$|z - z_0| = r$$

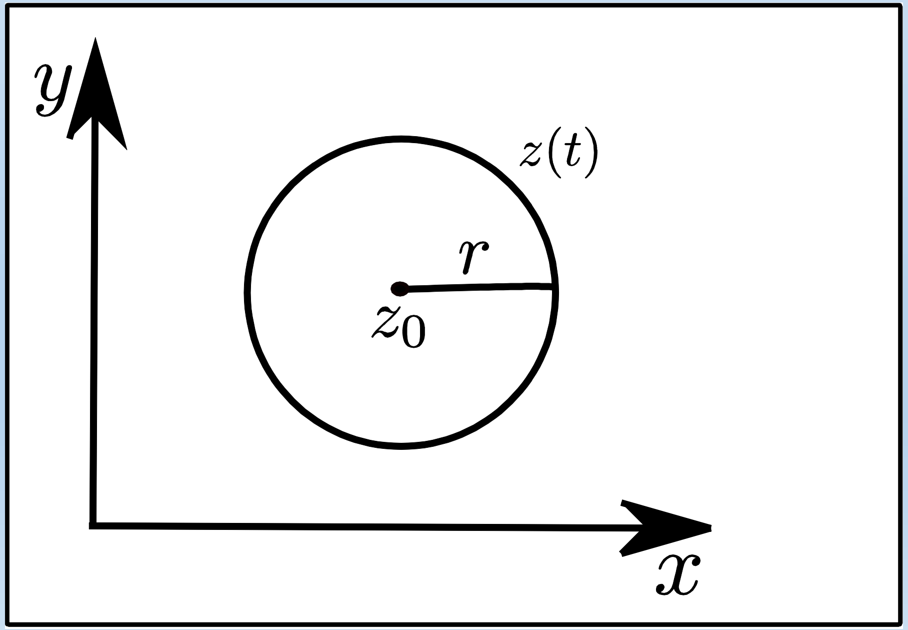
$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

$$x - x_0 = r\cos\theta$$

$$y - y_0 = r\sin\theta$$

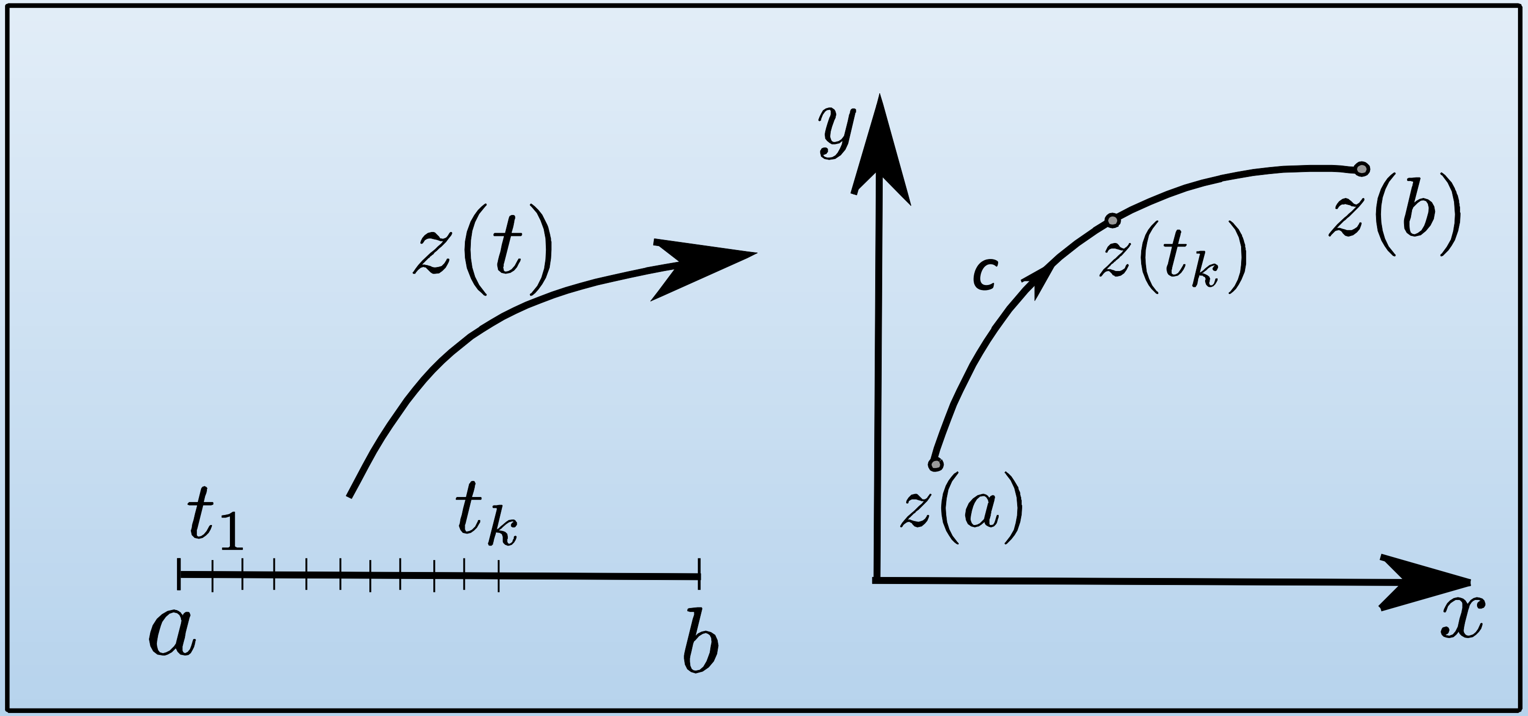
$$z = z_0 + r\cdot e^{i\theta}, \qquad -\pi \le \theta \le \pi$$



## Curves in the complex plane

**Def**. A curve is a range of continuous complex valued function defined in the real interval $a \le t \le b$




$$C : z(t) = x(t) + i \cdot y(t)$$

*z(t)*- parametrisation of curve ***C***

*z(a)* is called initial point and *z(b)* final point of curve ***C***

i) If *x(t)* and *y(t)* are differentiable, curve is smooth

i) A curve is simple if it does not cross itself

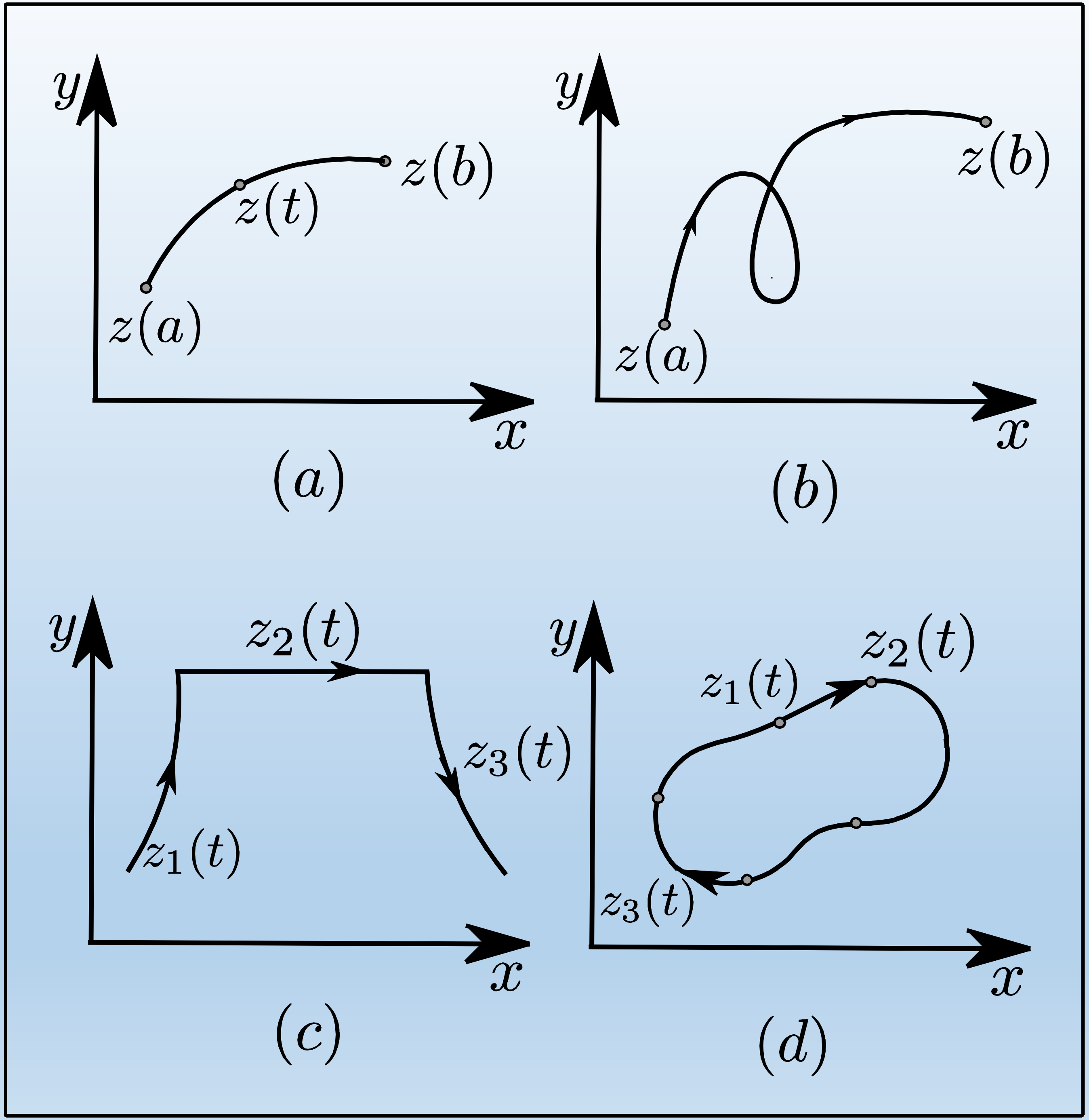
$$t_1 \ne t_2; \quad z(t_1) \ne z(t_2)$$

iii) A path is a finite collection of simple curves

$$\{z_1, z_2, \ldots z_n\}$$

such that a final point of $z_k$ coincides with initial point of $z_{k+1}$

iv) A countur is a path whose curves are smooth. When the initial point of $z_1$ coincides with the final point of $z_n$ the contour is simple closed conure.




Examples

a) Simple curve

b) Not simple curve

c) Path

d) Simple closed counture

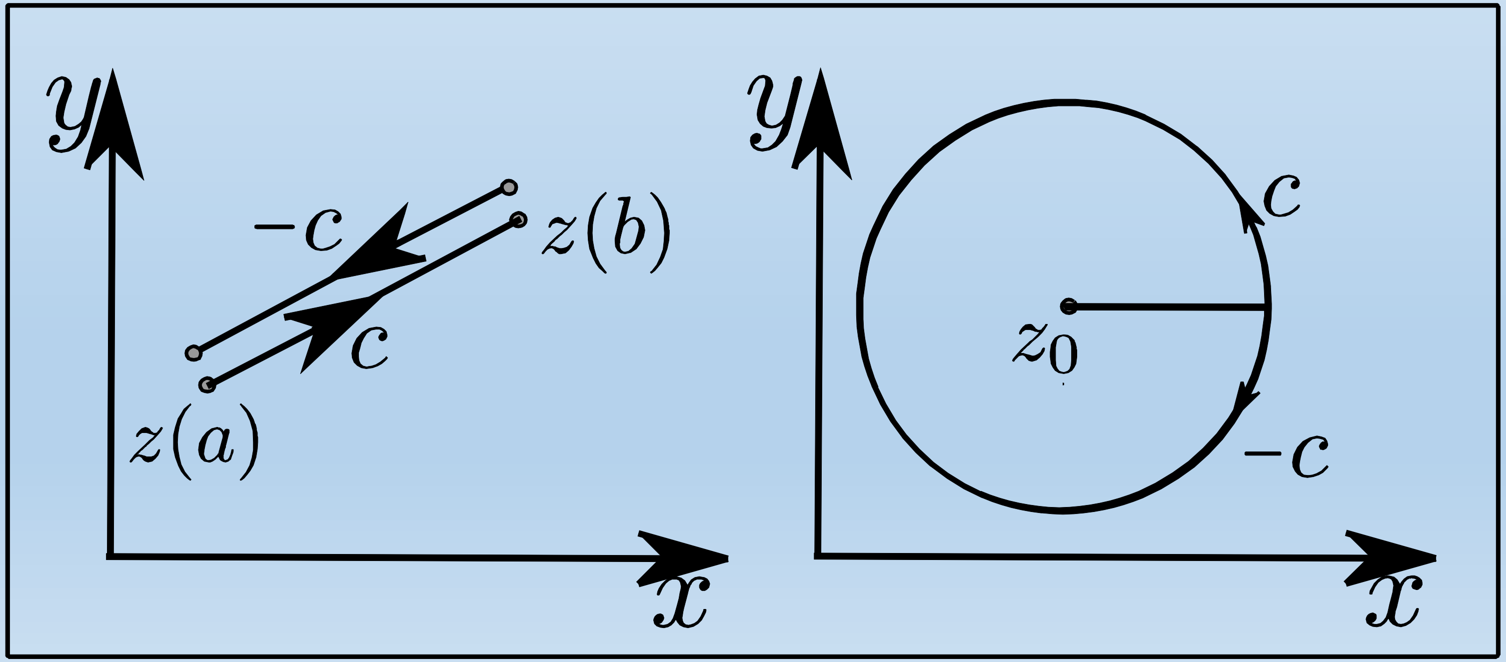
v) A curve is oriented. It goes from initial point ***z(a)*** to final point ***z(b)***.

vi) We define a curve **'– C'** as a range of another function $\gamma(t)$ having the same values as ***z(t)*** *but where initial and final values are reversed :*

$$C : z(t)$$

$$-C : \gamma(t) = z(a+b-t)$$

*vii) Any simple closed contour devides the complex plane into two domains. One is bounded and is called the* ***interior of C.*** The other is called exterior. Contour is positively oriented if the interior is on its left side ( counterclock wise orientation).



# Complex functions

***Def***. Complex function is a map $f: C \rightarrow C,\quad f(z) = w$
where both ***z*** and ***w*** are complex numbers , $z, w \in C$ .

Geometrically, ***f*** is correspondence between two complex planes, ***z*** and ***w***

$$f(z) = w(x, y) = u(x, y) + i \cdot v(x, y)$$

***u*** and ***v*** are real and imaginary parts of ***w***

Example:

$$f(z) = w = z^2$$

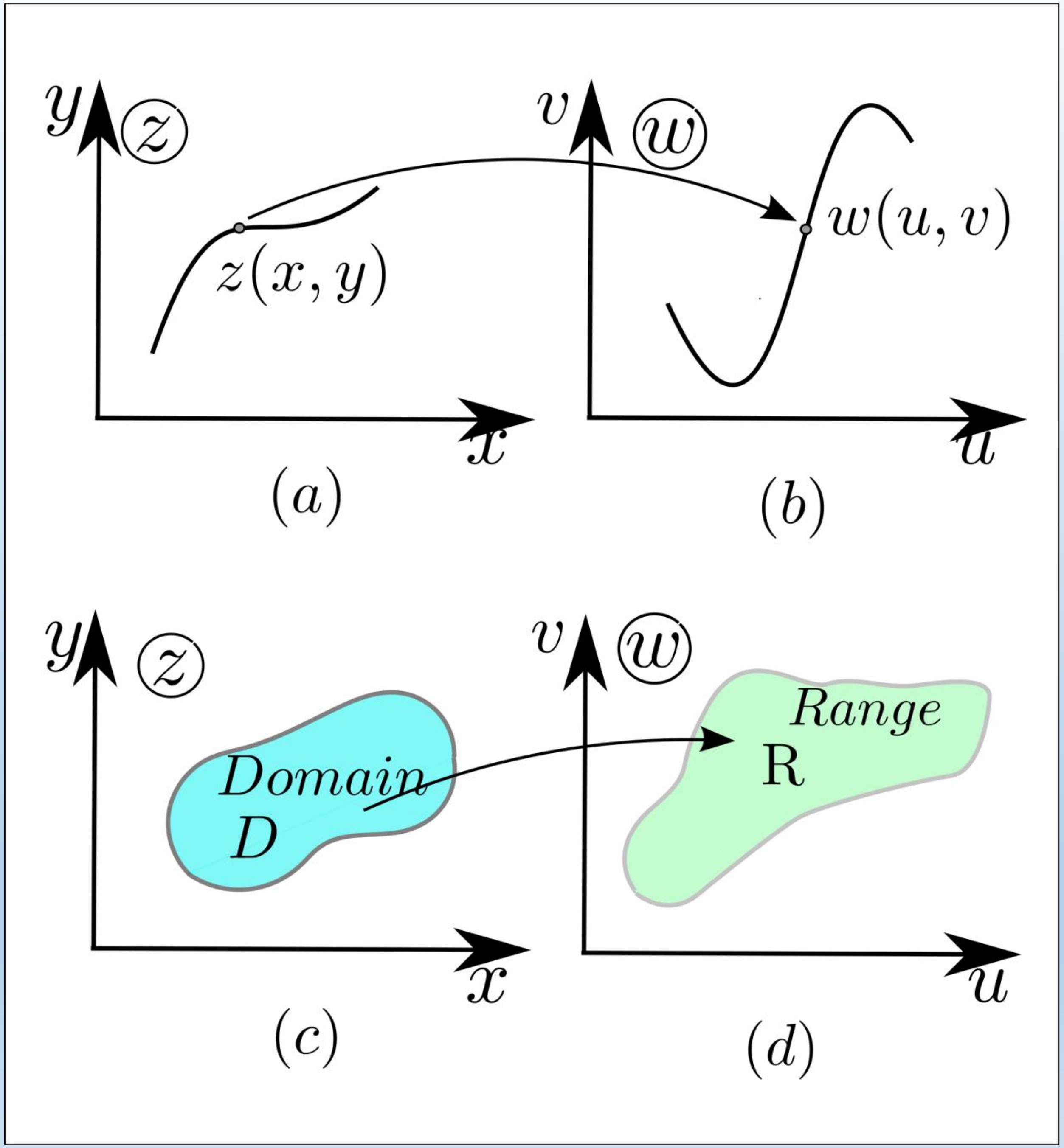
$$w = (x + iy)^2 = x^2 - y^2 + i \cdot 2xy$$

$$u = x^2 - y^2, \qquad v = 2xy$$

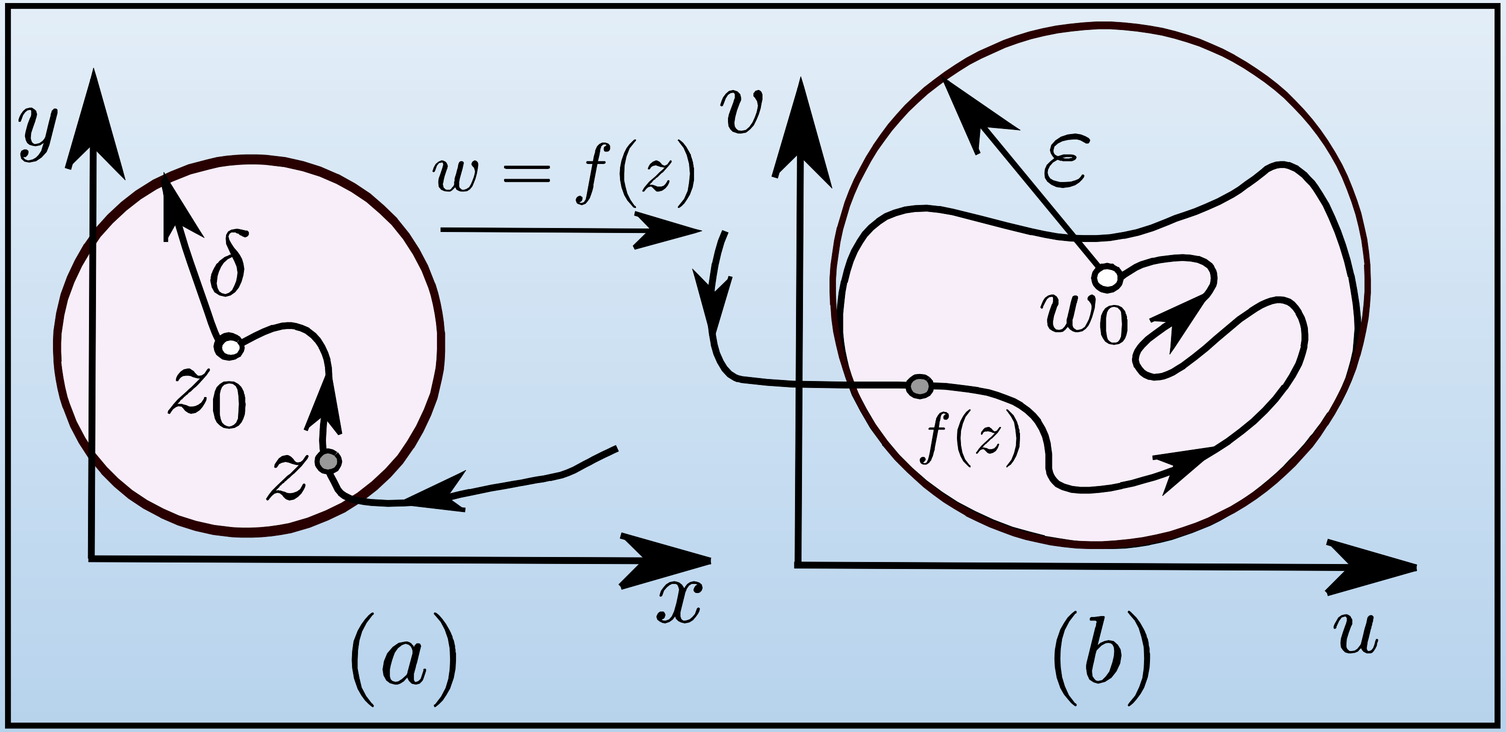
It maps for instance: ***y=mx*** to

$$u = (1 - m^2)x^2;\; v = 2mx^2$$

$$v = \frac{2m}{1 - m^2} u$$



# Limits of complex functions- continuity




Limits of complex functions are defined in terms of modula of complex numbers.
The expression

$$\lim_{z \to z_0} f(z) = w_0$$

means that for each ***real*** number **ε >0** there exists a ***real*** number **δ>0** such that:

$$|f(z) - w_0| < \varepsilon \text{ whenever } |z - z_0| < \delta$$

We say that function ***f(z)*** is continuous at $z = z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

In terms of functions ***u*** and ***v***

$$\lim_{z \to z_0} f(z) = u(x_0, y_0) + i \cdot v(x_0, y_0) = u_0 + i \cdot v_0$$

*iff*

$$\lim_{x,y \to x_0,y_0} u(x,y) = u_0, \quad \lim_{x,y \to x_0,y_0} v(x,y) = v_0$$

# Elementary functions

i) Polnomial function

$$P_n(z) = a_0 + a_1 z + \ldots + a_n z^n \qquad a_i, z \in C$$

ii) Rational function

$$w(z) = \frac{P_n(z)}{Q_m(z)}$$

iii) Exponential function

$$w(z) = e^z = e^x(\cos y + \sin y)$$

$$e^{2n\pi i} = 1$$

$$e^{z1} \cdot e^{z2} = e^{z1+z2}, \qquad \frac{e^{z1}}{e^{z2}} = e^{z1-z2}$$

iv) Trigonometric functions

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$tg\, z = \frac{\sin z}{\cos z}, \qquad ctg\, z = \frac{\cos z}{\sin z}$$

v) Hyperbolic functions

$$\sinh z = \frac{e^z - e^{-z}}{2}, \qquad \cosh z = \frac{e^z + e^{-z}}{2}$$

$$tgh\, z = \frac{\sinh z}{\cosh z}, \qquad ctgh = \frac{\cosh z}{\sinh z}$$

vi) Logarithmic function

$$\ln z = \ln\left(r \cdot e^{i\theta + 2n\pi i}\right) = \ln r + i\theta + 2n\pi i$$

Principal branch :

$$\ln z = \ln r + i\theta, \qquad -\pi < \theta \le \pi$$

vii) Invesre trigonometric functions

viii) Inverese hyperbolic functions

ix) Function

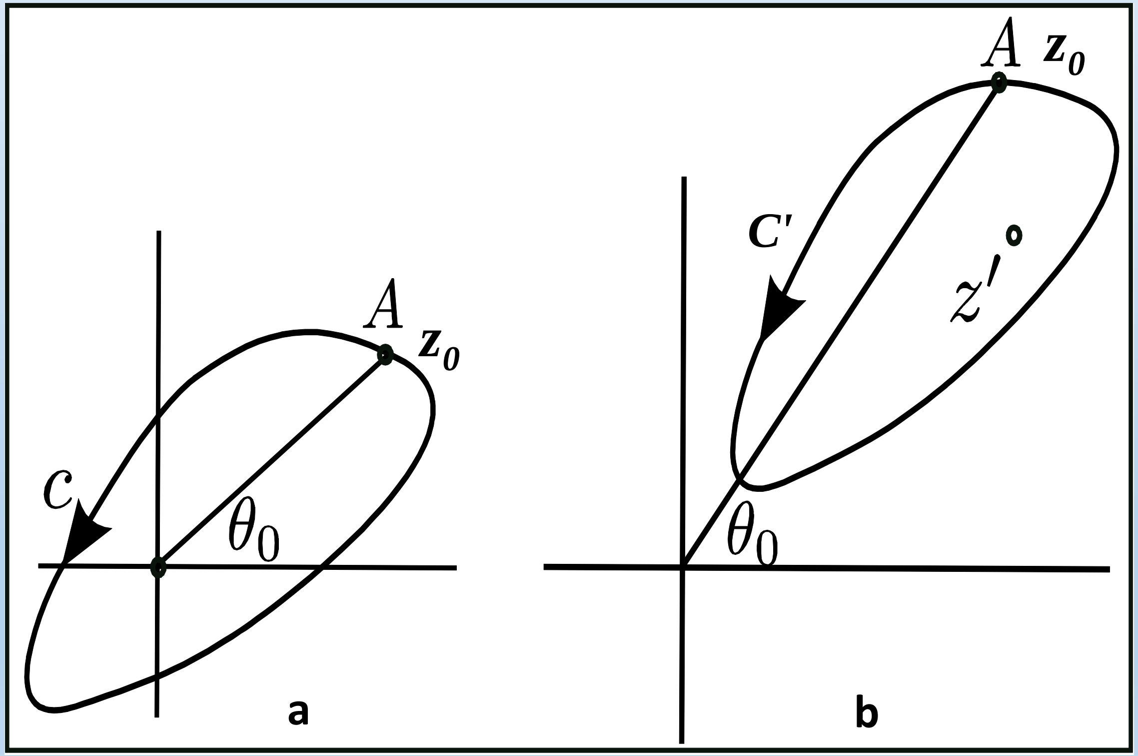
$$w(z) = z^\alpha = e^{\alpha \cdot \ln z}$$

# Multivalued functions, branch points, branch cuts….

Complex number in a polar form:

$$z = re^{i\arg z}, \qquad \arg z = \theta + 2n\pi$$

May lead to functions that can take different values at the same point in the complex plane- ***multivalued function***.




i) Consider function $f(z) = \sqrt{z} = f(r,\theta) = re^{i\frac{\theta}{2}}$

After making a complete circuit around ***C*** in fig. A

$$z(r_0,\theta_0) = z(r_0,\theta_0 + 2\pi)$$

$$f(r_0,\theta_0 + 2\pi) = \sqrt{r}e^{i\frac{\theta_0 + 2\pi}{2}} = \sqrt{r}e^{i\frac{\theta_0}{2}} \cdot e^{i\pi} =$$

$$= -\sqrt{r}e^{i\frac{\theta_0}{2}} = -f(r_0,\theta_0)$$

$$f(r_0,\theta_0 + 2\pi) \neq f(r_0,\theta_0)$$

ii) Consider function :

$$f(z) = \ln z = \ln(re^{i\arg z})$$

$$f(z) = \ln r + i\arg z = \ln r + i\theta + i2n\pi$$

Encircling ***z=0*** arround ***C*** starting at point $z_0$:

$$z_0(r_0,\theta + 2\pi) = z_0(r_0,\theta)$$

$$\ln(z_0(r_0,\theta + 2\pi)) = \ln r + i\theta + 2\pi i$$

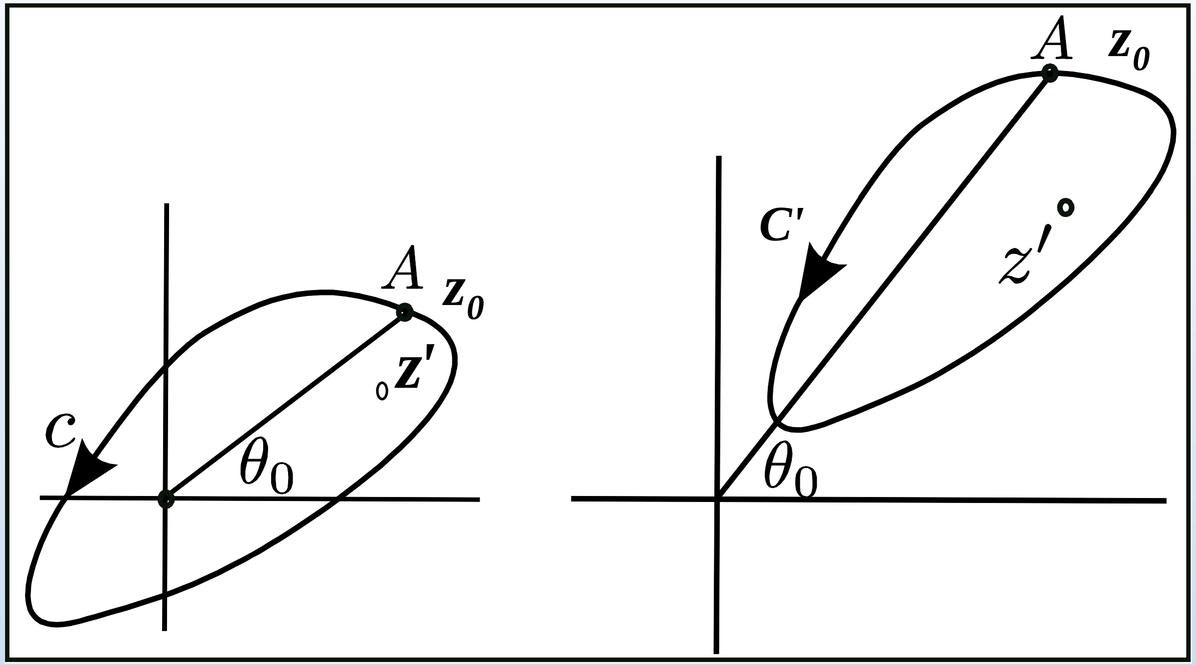
$$\ln(z_0(r_0,\theta)) = \ln r + i\theta$$

$$\ln(z_0(r_0,\theta + 2\pi)) - \ln(z_0(r_0,\theta)) = 2\pi i$$

What is the difference between the contours in these two figures which makes the behaviousr of $\sqrt{z}$ and ***lnz*** so different ?

***Answer:*** The first contour encloses the erigin ***z=0*** which the second does not.

***The origin is a branch point of functions*** $\sqrt{z}$ ***and ln z***




Def. The point $z_b$ is called a branch point for the complex multivalued function $f(z)$ if the value of $f(z)$ does not return to its initial value as a closed curve around $z_b$ is traced (starting at some arbitrary point on the curve).

Important : What ***maters*** in def. of branch point is the local behavour of function $f(z)$ near $z_b$. For example, consider $ln\,(z)$, take a point $z'$ and a contour $C$ around it (that also enclose $z = 0$ ). The value of $ln(z)$ will change as this curve is traced but $z'$ is not a branch point of $ln\ (z.)$. For contour $C'$ close to $z'$ there is no change of $ln\,(z)$.

Hence, point $z'$ is not a branch point of function $ln(z)$.

Function $\ln(z)$ is instructive example of mutivalued function.

Studying behaviour of $\ln(1/z) = -\ln(z)$ around $z = 0$, shows that infinity is also a branch point of function $f(z)$.

Function $\ln(z)$ has two branch points : $z = 0$ and $z = \infty$.

This is a general situation - functions have no sole branch point.

Branch points always appear in pairs.

How to obtain a single valued function out of multtivalued one?

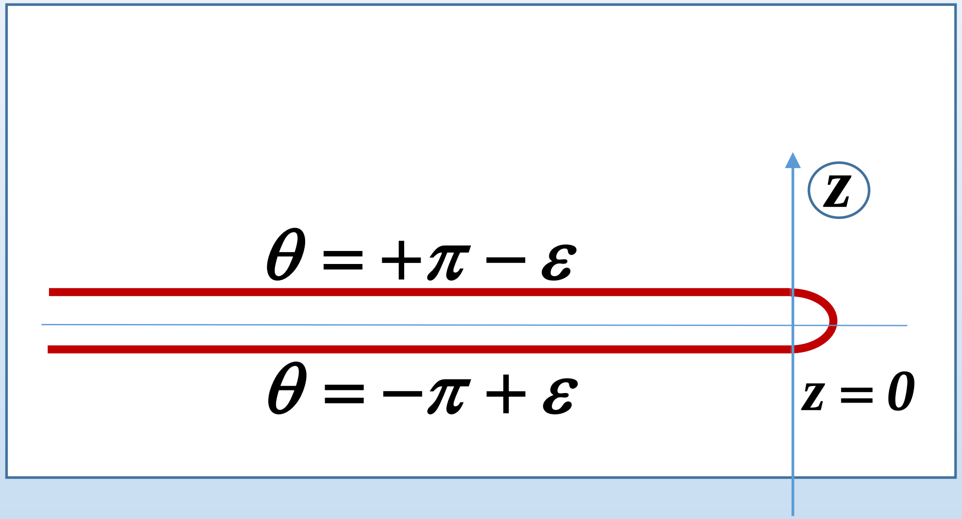
Prevent encircling of branch points!

To do it, one introduces a branch cut, a line connecting branch points and agree never to cross it.

Branch cut for a function $\ln(z)$ is shown in a figure.

Function $\sqrt{z}$ has the same cut.

Branch points always appear in pairs




Branch cut

Discontinuity of function $\ln z$ across the cut.

$$Dis = [\ln r + i(\pi - \varepsilon)] - [\ln r + i(-\pi + \varepsilon)] = 2i\pi - 2i\varepsilon$$

$$\lim_{\varepsilon \to 0} Dis = 2i\pi = 2i\, Im(\ln r + i\pi)$$

**!**

It is to point out that a branch points of an function are unique.
Branch cuts are not. A cut along any path preventing encircling any of branch points is allowed.

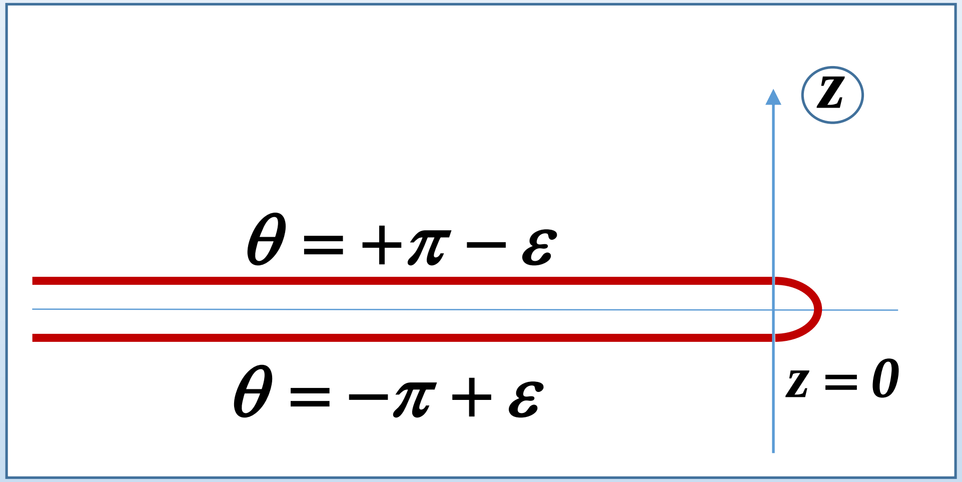
***Branches of multivalued functions***

Let's define a set of functions

$$f_n(z) = f_n(r,\theta r = \ln r + i\theta + 2ni\pi, \quad -\pi < \theta < \pi, \quad n = 0,1,...$$

Observe :

$$f_{n+1}(r, -\pi + \varepsilon) = \ln r - i\pi + i\varepsilon + 2(n+1)i\pi$$
$$= \ln r + i\varepsilon + (2n+1)i\pi$$
$$f_n(r, +\pi - \varepsilon) = \ln r + i\pi - i\varepsilon + 2ni\pi$$
$$= \ln r - i\pi + (2n+1)i\pi$$




$$\lim_{\varepsilon \to 0} f_{n+1}(r, -\pi + \varepsilon) = \lim_{\varepsilon \to 0} f_n\ (r, +\pi - \varepsilon)$$

Value of $f_n$ just above the cut is equal to value of $f_{n+1}$ just below the cut!

This leads to idea of Riemann surfaces.

***Riemann surfaces ( for $\ln z$ )***

i) Superpose an infinite number of cut complex planes one on top of the other, each plane corresponding to different value of n.

ii) Connect adjecent planes along a cut in such a way that the lower edge of the $n_{th}$ plane is connected to upper edge of $(n-1)_{th}$ plane.

All planes contain two branch points

***In this construction, if we cross a cut we end up on a different plane.***

The surface constructed in such a way is called Riemann surface.

Each plane is called ***Riemann sheet.***

A single valued function defined on a given Riemann sheet is called a ***branch*** of original multivalued function.

In applications:

i) Define branch points of a given function

ii) Cut the plane to avoid encircling of the branch points

iii) Specify branch- single valued function

For functions $\ln(z)$ and $\sqrt{z}$ principal barnches are defined as

$$\ln(z) = \ln r + i\theta \qquad -\pi < \theta < \pi$$

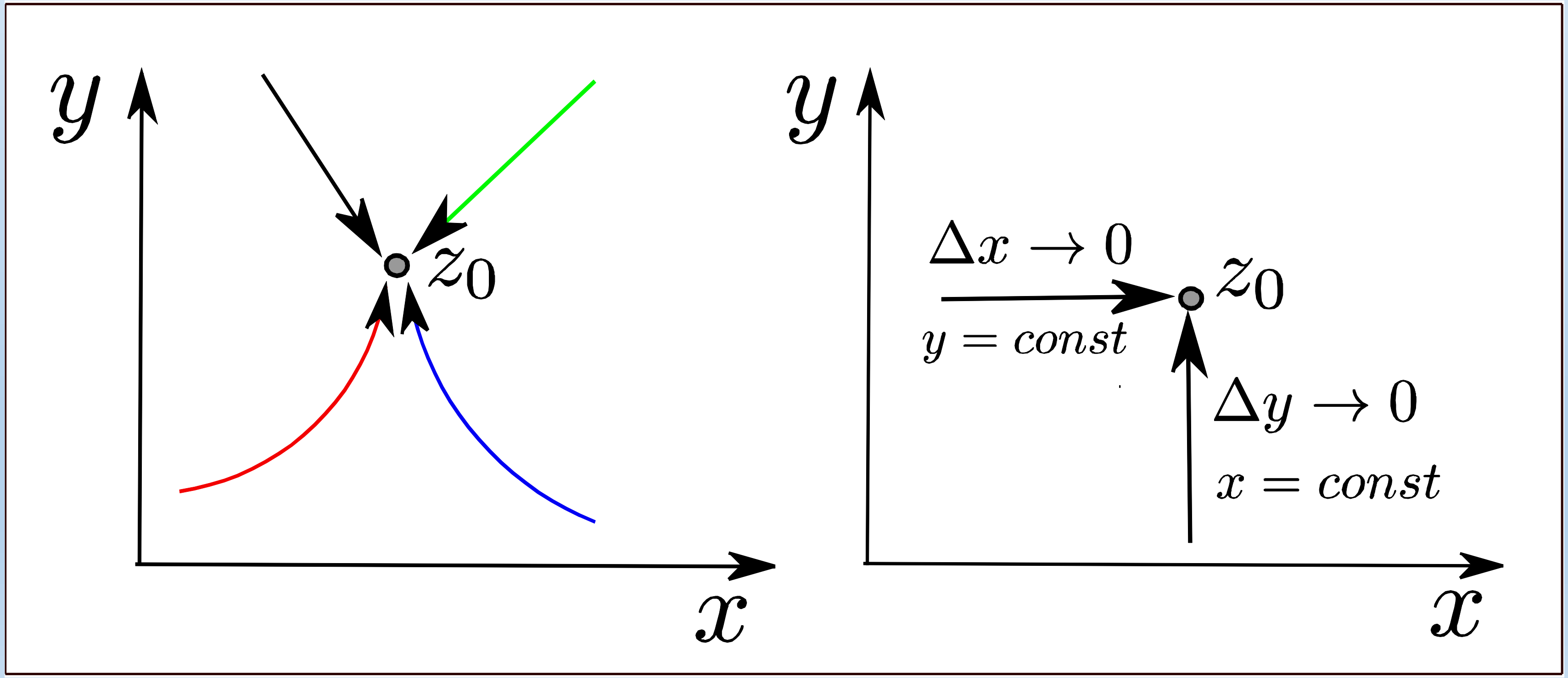
$$\sqrt{z} = r^{1/2} e^{i\theta/2}, \qquad -\pi < \theta < \pi$$

# Derivative of complex function

Def. Derivative of a complex function f(z) at $z = z_0$ is:

$$f'(z_0) = \left.\frac{df(z)}{dz}\right|_{z=z_0} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Provided that the ***limit exists*** and is ***independent of Δz***



Example 1.

Check differentiability of function $f(z) = x^2 + 2i \cdot y^2$ at point *z=1+i*

*Let's start from the definition:*

$$\left.\frac{df(z)}{dz}\right|_{z=1+i} = \lim_{\Delta z \to 0} \frac{f(1+i+\Delta z) - f(1+i)}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{2\Delta x + 4i\Delta y + (\Delta x)^2 + 2i(\Delta y)^2}{\Delta x + i\Delta y}$$

Let's approach z=1+i along the line :

$$y = 1 + m(x-1), \quad \Delta y = m \cdot \Delta x$$

*then*

$$\left.\frac{df(z)}{dz}\right|_{z=1+i} = \frac{2+4im}{1+im}$$

We obtain infinitely many values of derivative depending on *m*.

Conclusion: Derivative of function $f(z) = x^2 + 2i \cdot y^2$ does not exist at the point *z=1+i.*

Example 2.

Using the same way of approaching z=1+i, show that function

$$f(z) = z^* = x - i \cdot y$$

does not have derivative at that point ( Even more: has not at any point !).

!!

Differentiability puts severe restrictions on complex functions

## Cauchy- Riemann conditions

Q: Are there any criteria which may tell us if a given complex function is differentiable at a given point?

A: Yes.

The function $f(z) = u(x,y) + i \cdot v(x,y)$

is differentiable at a given point in the complex plane iff the Cauchy- Riemann conditions

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y}, \quad \frac{\partial u(x,y)}{\partial y} = -\frac{\partial v(x,y)}{\partial x}$$

are satisfied and all partial derivatives of *u* and *v are continues.*

*In that case:*

$$\frac{df(z)}{dz} = \frac{\partial u(x,y)}{\partial x} + i\frac{\partial v(x,y)}{\partial y} = \frac{\partial v(x,y)}{\partial y} - i\frac{\partial u(x,y)}{\partial x}$$

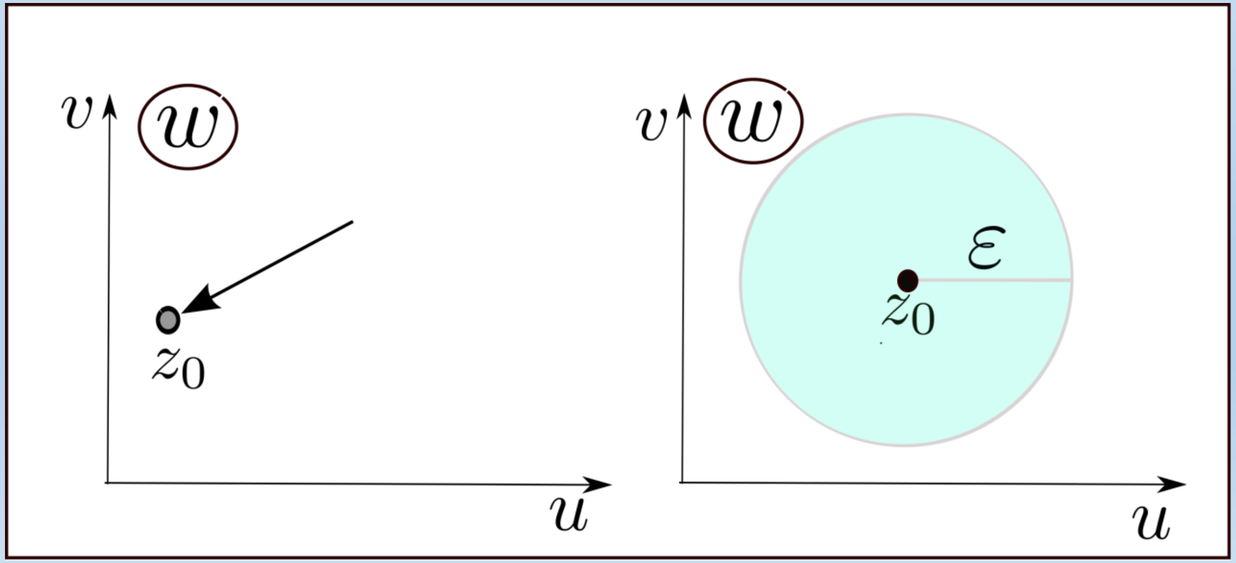
## Analytic function

We are seldom interested in studying functions that are or are not differentiable at a given point. Complex functions that have a derivatives at all points in a neighborhood of a given point $z_0$ deserve a detailed study.

**Def.** A function $f : C \to C$ is called ***analytic*** at $z_0$ if it is differentiable ***at*** $z_0$ and at all points in ***some neighborhood*** of $z_0$ .

A point at which ***f(z)*** is analytic is called regular point of ***f***.

A point at which ***f*** is not analytic is called a singular point or a singularity of ***f.***
If function ***f*** is analytic at each point in the region $R \subset C$, we say that function ***f*** is analytic in ***R .***



Formal rules for differentiation for real functions may be applyied to complex functions.

i) Let f and g be analytic in some region $R \subset C$ ***and*** $\lambda \in$ ***C***

Then: ***λf***, ***f+g***, and ***f ·g*** are analytic functions and:

$$[\lambda f(z)]' = \lambda \cdot f'(z), \qquad [(f(z)+g(z)]' = f'(z)+g'(z)$$

$$[f(z)\cdot g(z)]' = f'(z)\cdot g(z) + f(z)\cdot g'(z)$$

Consequence: As ***f(z) =z*** is analytic function in ***C***, then any polynomial

$$P_n(z) = \sum_{n=1}^{N} c_n z^n$$

is analytic in ***C.***

ii) Let ***f*** be analytic in ***R*** then, providing that ***f(z)≠0***, then ***1/f(z)*** is analytic function and its derivative is given by:

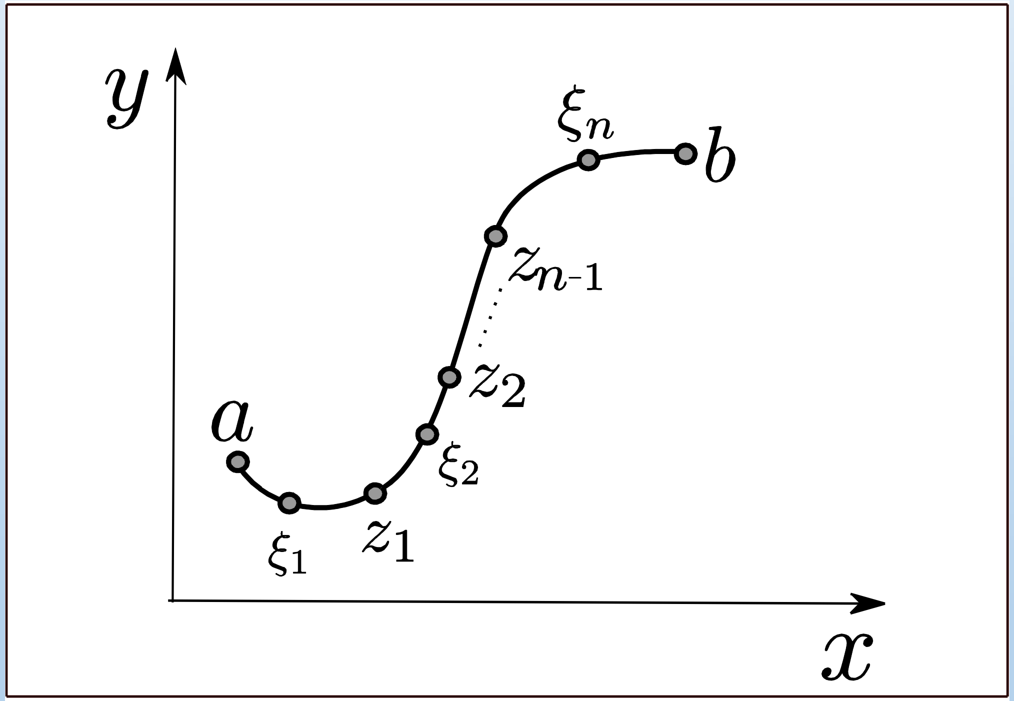
$$\left[\frac{1}{f(z)}\right]' = -\frac{f'(z)}{f(z)^2}$$

Consequence: Any rational function $f(z) = \frac{P_n(z)}{Q_n(z)}$ is analytic in ***C*** except points where

$Q_n(z) = 0$

# Contour integrals

Let complex function f(z) be continuous along a smooth curve C.




For each arc joining $z_k$ and $z_{k+1}$ choice one point $\xi_k$ and form a sum:

$$S_n = \sum_{k=1}^{n} f(\xi_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} f(\xi_k)\cdot \Delta z_k$$

Quantity

$$\lim_{n\to\infty} S_n = \int_C f(z)\cdot dz$$

is called complex line integral or line integral of ***f(z)*** along curve ***C***.

$$f(z) = u(x,y) + i\cdot v(x,y)$$

$$\int_C f(z)dz = \int_C \big(u(x,y) + i\cdot v(x,y)\big)\cdot\big(dx + idy\big)$$

$$= \int_C u(x,y)dx - v(x,y)dy + i\int_C v(x,y)dx + u(x,y)dy$$

From definition it is clear that integration along curve '***-C***' gives result of oposite sign:

$$\int_{-C} f(z)dz = -\int_C f(z)dz$$

The integral $\int_C f(z)dz$ may be cast in another form:

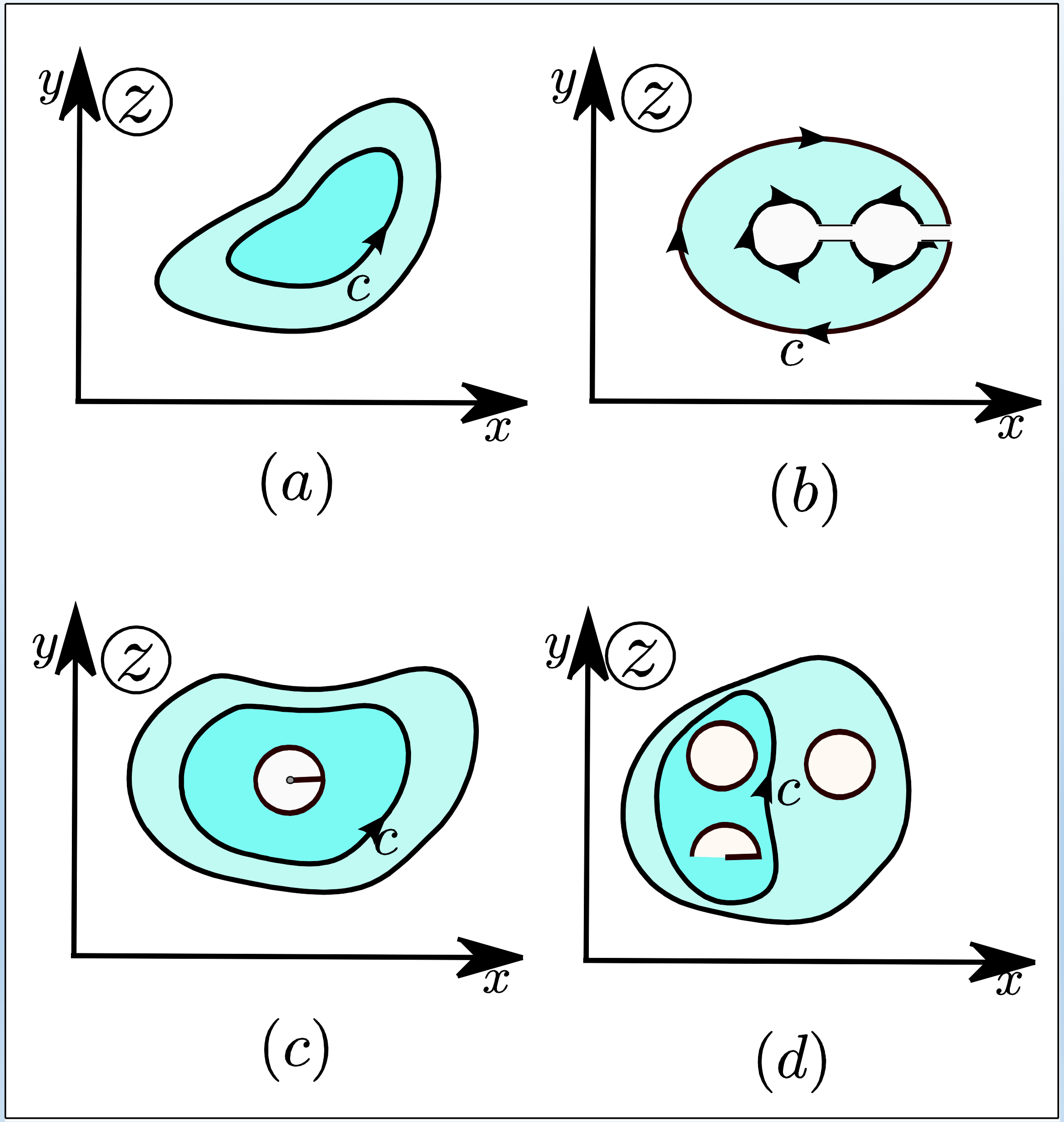
$$S_n = \sum_{k=1}^{n} f(\xi_k)\frac{z_k - z_{k-1}}{t_k - t_{k-1}} \cdot (t_k - t_{k-1}) = \sum_{k=1}^{n} f(\xi_k)\frac{\Delta z_k}{\Delta t_k} \cdot \Delta t_k$$

$$\lim_{n\to\infty} S_n = \int_C f(z)\,dz = \int_a^b f((z(t))z'(t)\,dt$$

Obtained formula reduce calculation of contour integral to calculation of integral of complex function over real interval ***a≤ t≤ b***

# Simply and multiply connected regions

**Def.** A region ***R*** is called simply connected if any simple closed conture which lies in ***R*** can be shrank to a point without leaving ***R***. A region ***R*** which is not simply connected is called multiply connected.

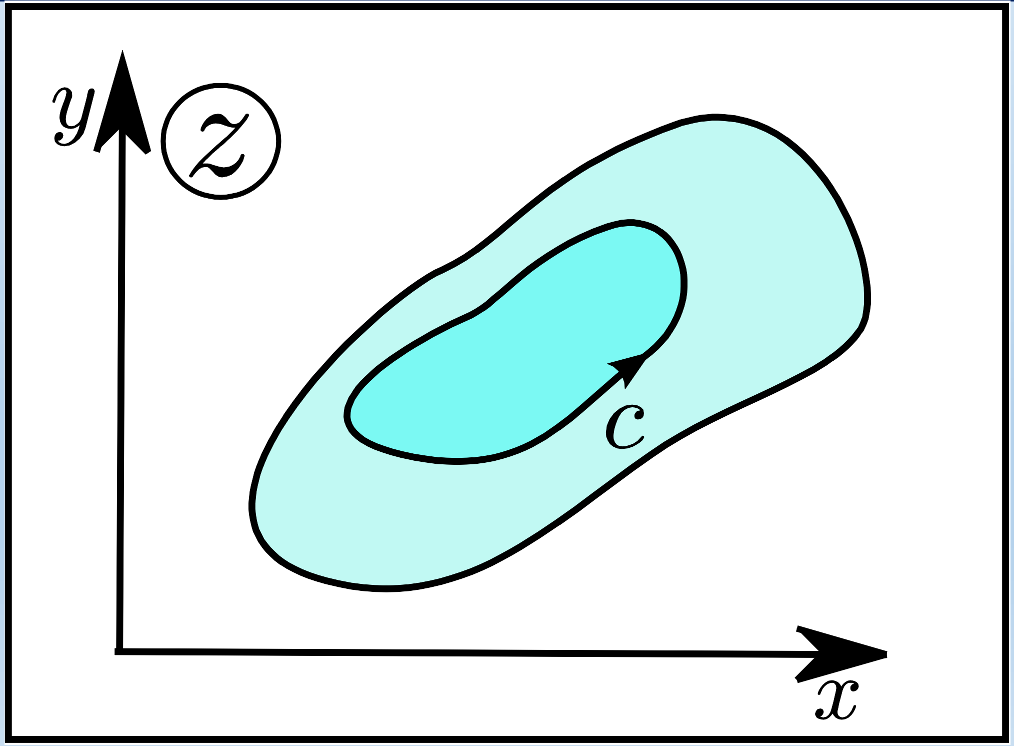


# Cauchy's theorem

Cauchy' s theorem, also known as Cauchy- Goursat theorem is the most important theorem in complex analysis.

Tm. Let *f(z)* be ***analytic*** on a simply closed contour ***C*** and in ***all*** points inside ***C***.
*Then:*

$$\int_C f(z)\,dz = 0$$

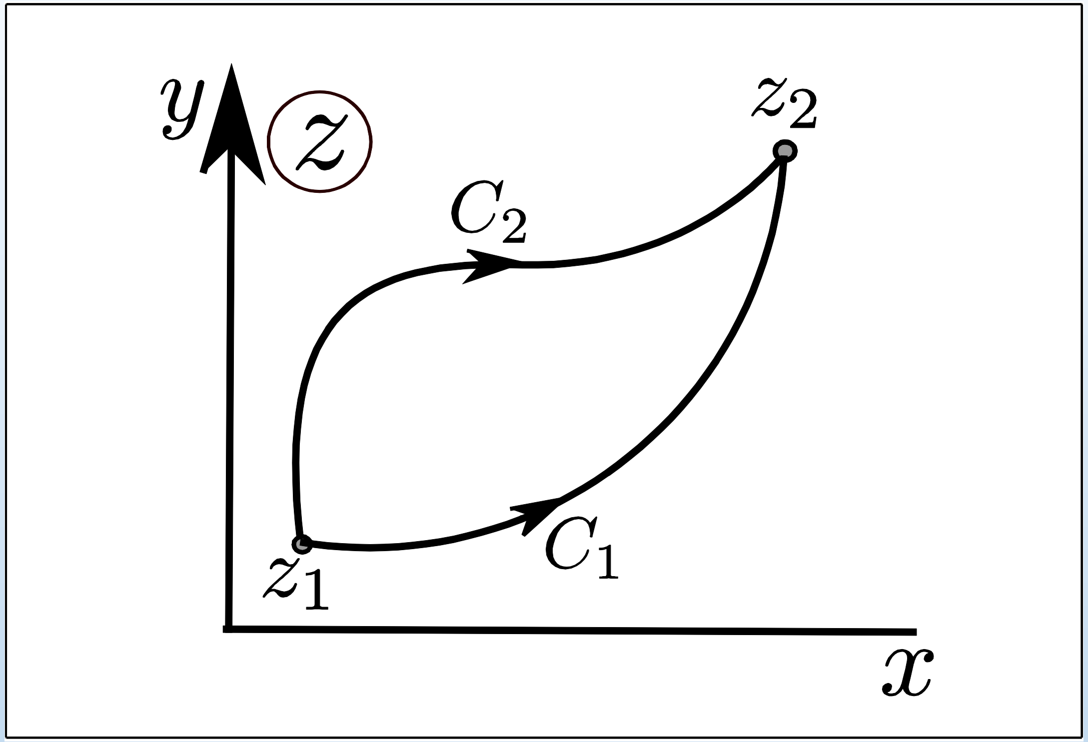


## Simple Direct Consequencies

i) If ***f (z)*** is ***analytic*** in a symple connected region ***R***, then the integral:

$$I = \int_{z_1}^{z_2} f(z)dz$$

Does not depend of the path in ***R*** joining points $z_1$ and $z_2$.




Example:

Calculate integral

$$I = \int_C z dz$$

Along curves:

$$C_1 : z(t) = t + 2i \cdot t, \qquad 0 \le t \le 1$$

$$C_2 : z(t) = t + 2i \cdot t^2, \qquad 0 \le t \le 1$$

$$C_3 : z(t) = \begin{cases} t, & 0 \le t \le 1 \\ 1 + i \cdot (t-1), & 0 \le t \le 1 \end{cases}$$

ii) **Deformation of contour**

Let $C_1$ and $C_2$ are two simple positively oriented contours such that $C_1$ lies interior to $C_2$.
If f(z) is analytic in a region ***R*** containing both contours than:

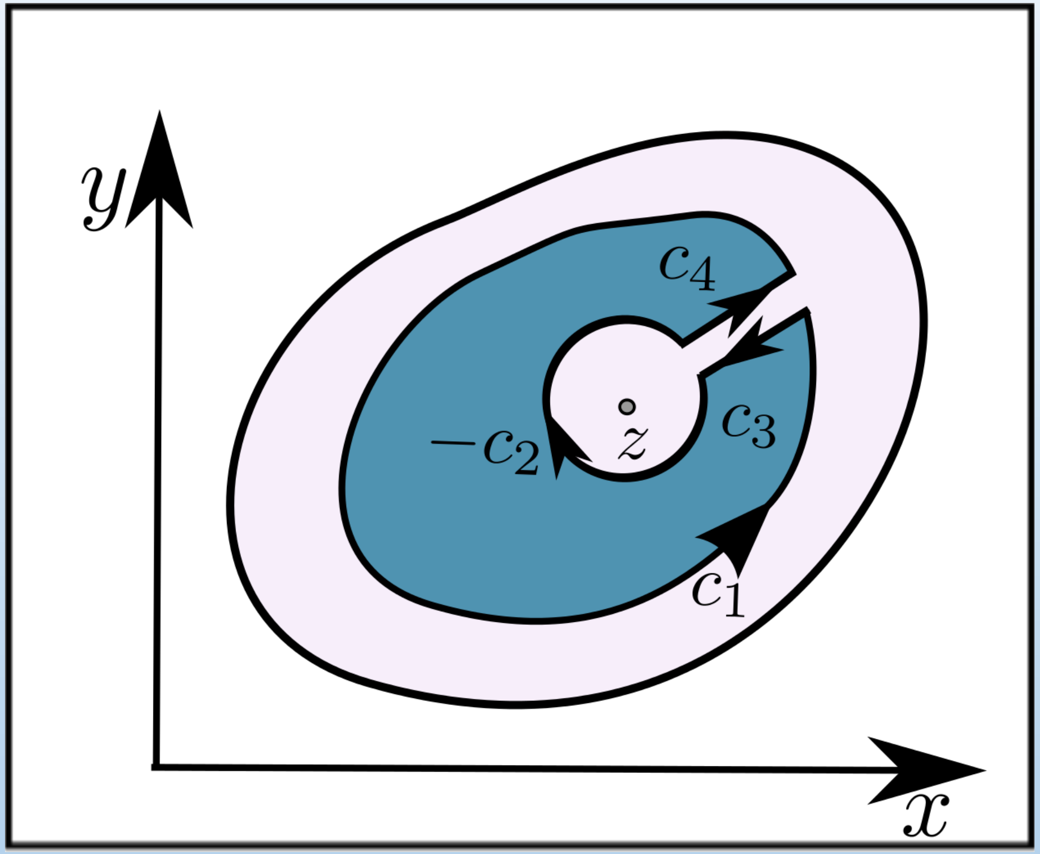
$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Complicated contours may be replaced by simpler one.

$$\int_{C} f(z)\,dz = \int_{C_1} f(z)\,dz + \left\{ \int_{C_3} f(z)\,dz + \int_{C_4} f(z)\,dz \right\} - \int_{C_2} f(z)\,dz = 0$$

$$\int_{C_2} f(z)\,dz = \int_{C_1} f(z)\,dz$$

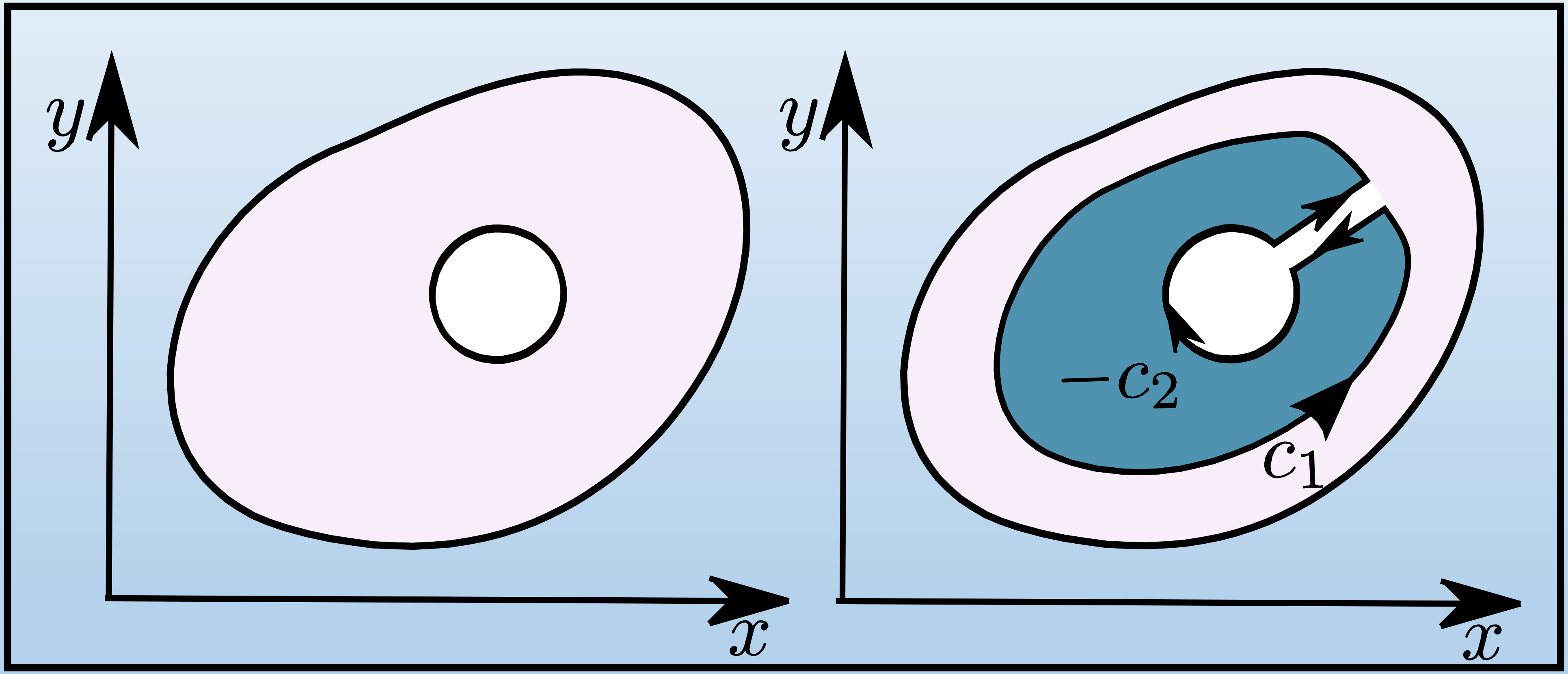
Integrals along cross cut cancel out.



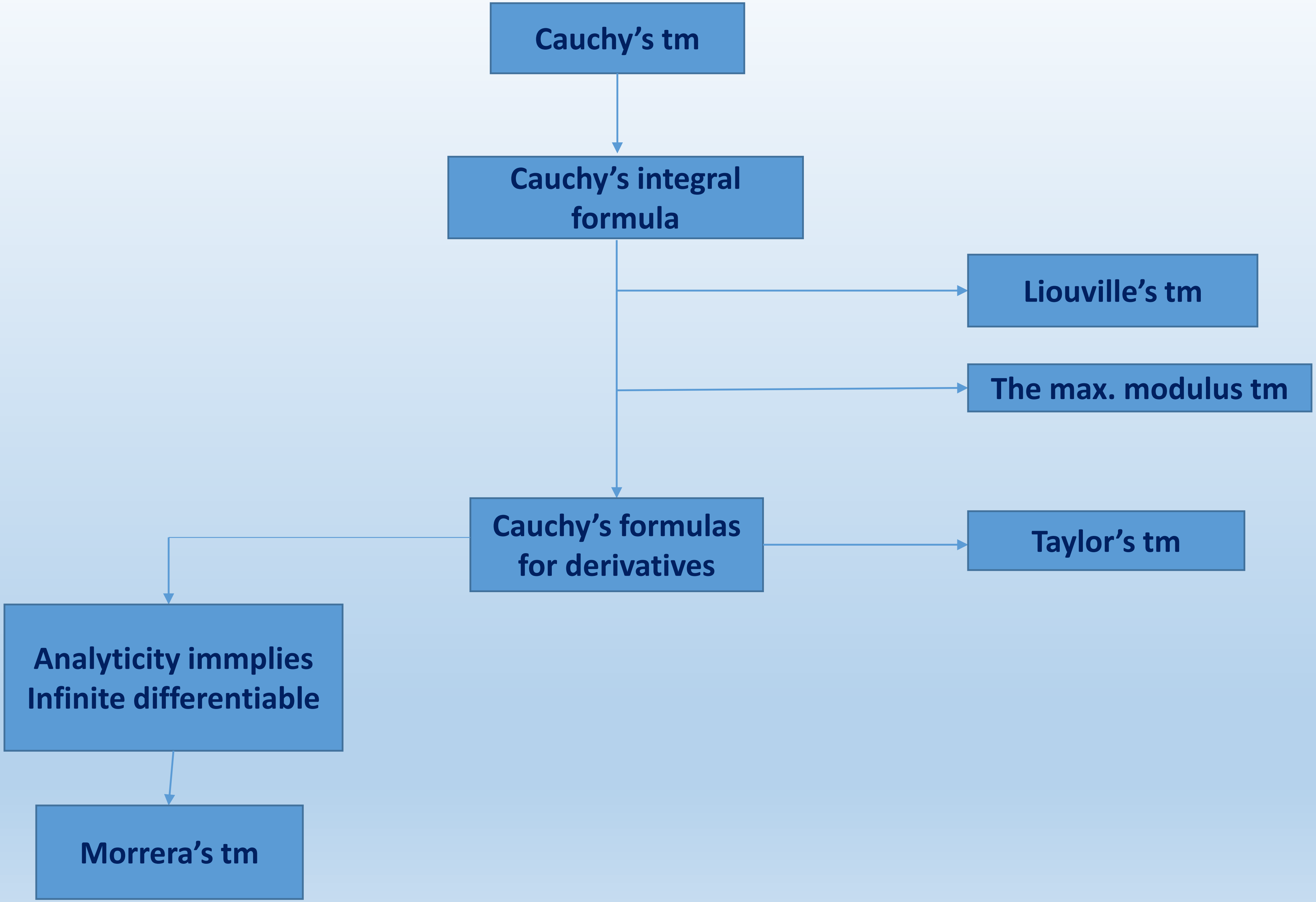

**Whatever you do, you do the same !**

## Cauchy-Goursat theorem for multiply connected regions

Multyply connected region may be reduced to simply connected one using cross cut(s) as shown in a drawing.
As in previous case, integrals along cross cut cancel out.




$$\int_C f(z)dz = \int_{C1} f(z)dz - \int_{C2} f(z)dz = 0$$


Cauchy’s tm
Cauchy’s integral formula
Liouville’s tm
The max. modulus tm
Cauchy’s formulas for derivatives
Taylor’s tm
Analyticity immplies Infinite differentiable
Morrera’s tm

## Cauchy's integral formula

Amazing formula: Gives value of an analytic function in every point inside a simple closed contour when its value on the contour is given.

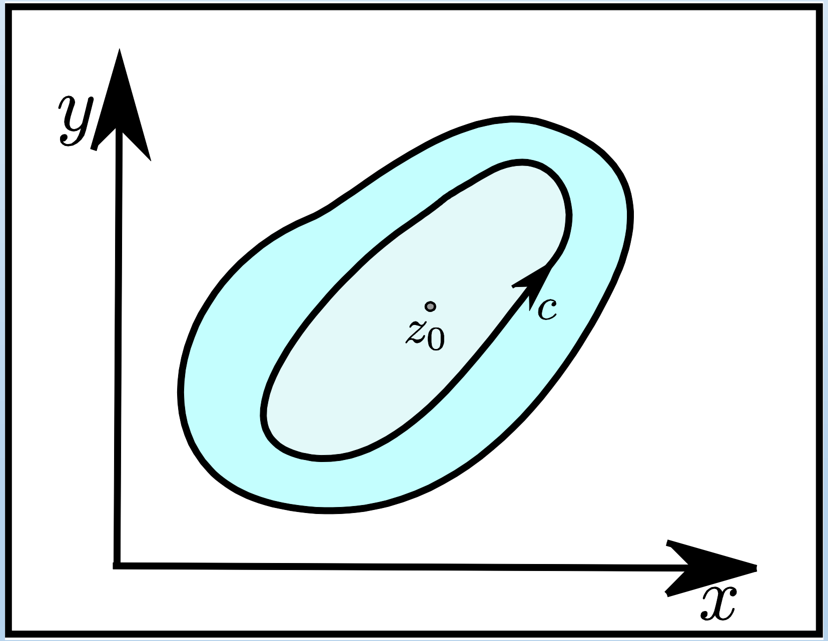
An analytic function is not free to change inside a region once its values are fixed on the contour enclosing that region.

**Tm**. Let ***f(z)*** be analytic function on and inside a positively oriented contour ***C.***

Than, if $z_0$ is inside contour:

$$f(z_0) = \frac{1}{2\pi \cdot i} \int_C \frac{f(z)dz}{z - z_0}$$

if $z_0$ is outside contour then $f(z_0) = 0$ .



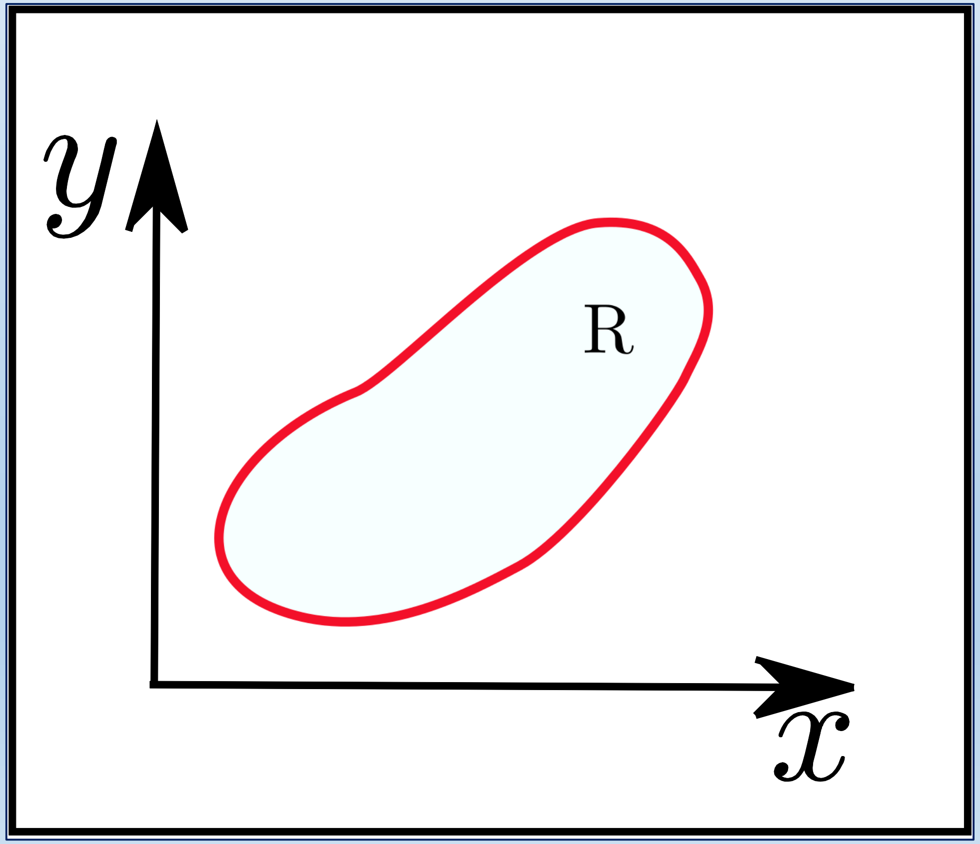
## Liouville's theorem

**Def.** Function is bounded in a region ***R*** if there is a constant ***M*** such that $|f(z)| \leq M$

**Tm**. Let ***f(z)*** be analytic and bounded in the whole complex plane ***C.***
Than function ***f(z)*** is constant

## The maximum modulus theorem

**Tm.** The absolute value of an analytic function ***f(z)*** can not have a local maximum within a region ***R*** of analyticity of the function. Maximum can be achived only on the border of ***R.***



## Cauchy's formula for derivatives

**Tm**. Let ***f(z)*** be analytic inside and on the boundary ***C*** of a simple connected region ***R.***

Derivatives of ***all orders*** of function ***f(z)*** exist in a region ***R*** and are ***themselves analytic functions in the same region***

The **n-th** derivative is given by:

$$f^{(n)}(z) = \frac{n!}{2\pi \cdot i} \int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}}$$

!

Reminder: We defined **analytic function** in a region ***R*** as a complex function having a **first derivative**.

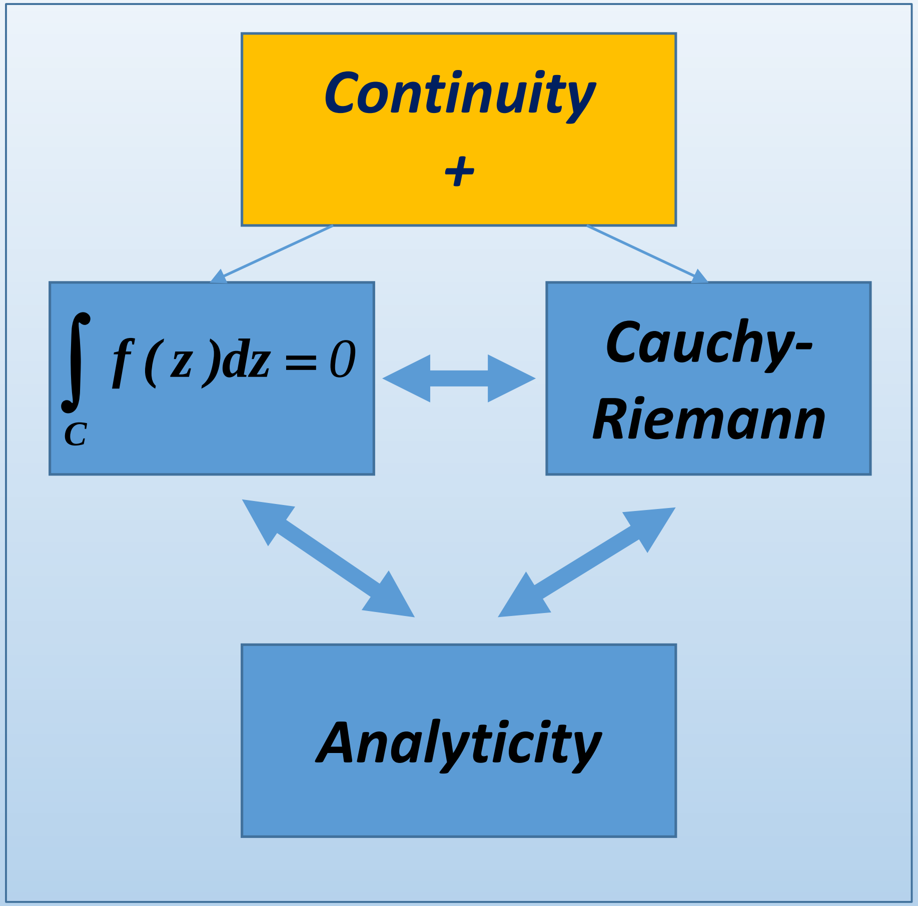
***Theorem claims that, being analytic, function is infinitely derivable.***

# Morrera's theorem

**Tm.** Let function ***f(z)*** be continuous in a simple connected region ***R.***
If for each simple closed contour ***C*** in ***R*** holds***:***

$$\int_C f(z)dz = 0$$

then ***f*** is analytic in ***R.***




Morrera's TM. Is converse of Cauchy's formula. It serves as a tool to identify analytic function in a given simple connected region. It is also an integral analog of Cauchy- Riemann relations.

## Digression- Power series

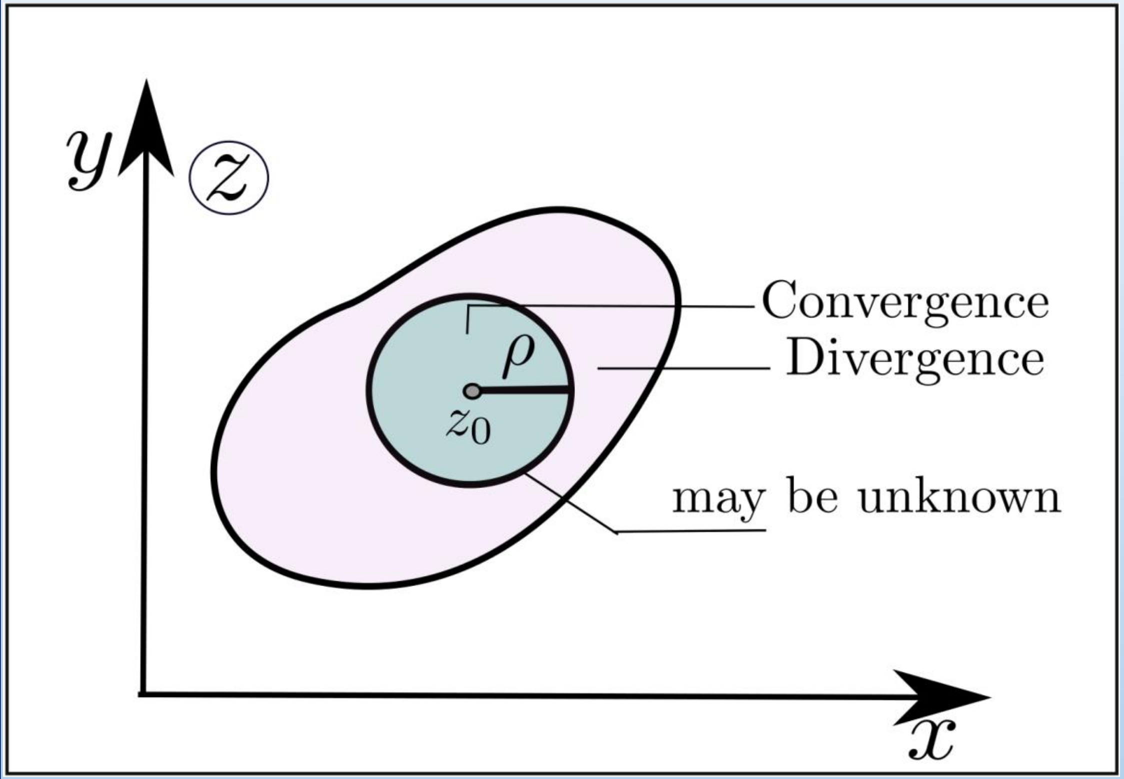
i) A function

$$S(z)=\sum_{n=0}^{\infty} c_n (z-z_0)^n$$

is called power series.
Power series converges if $|z-z_0|<\rho$ and diverges if $|z-z_0|>\rho$.
We call the number $\boldsymbol{\rho}$ radius of convergence of the power series.

*ii)* A power series is said to converge absolutely if the real series

$$S(z)=\sum_{n=0}^{\infty} |c_n||(z-z_0)^n|$$ converges.



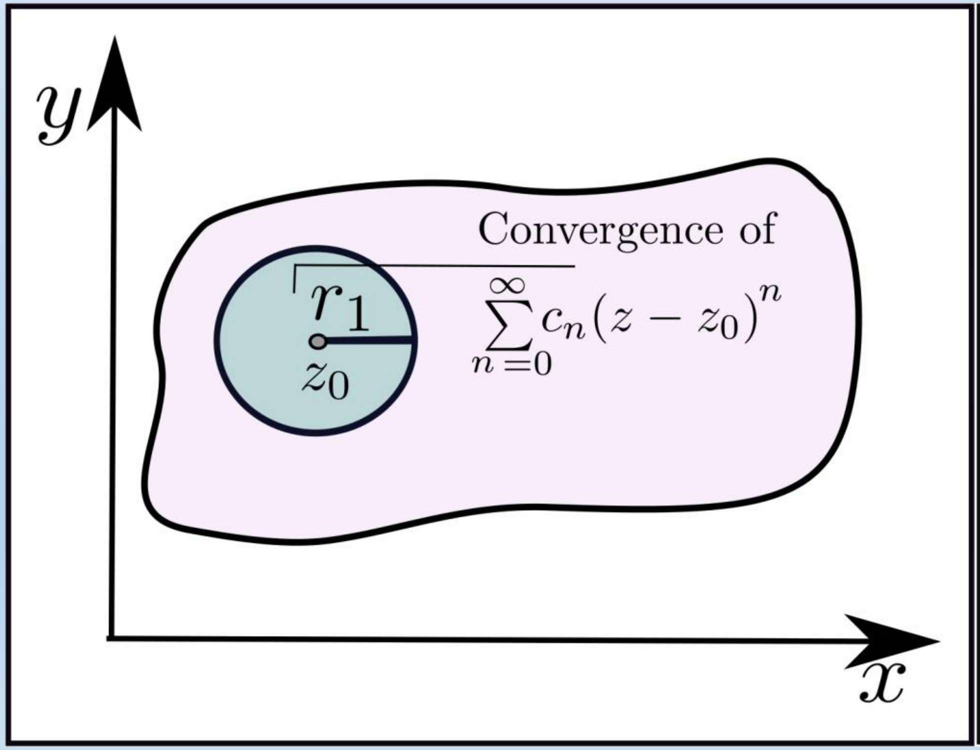
# Digression –Power series

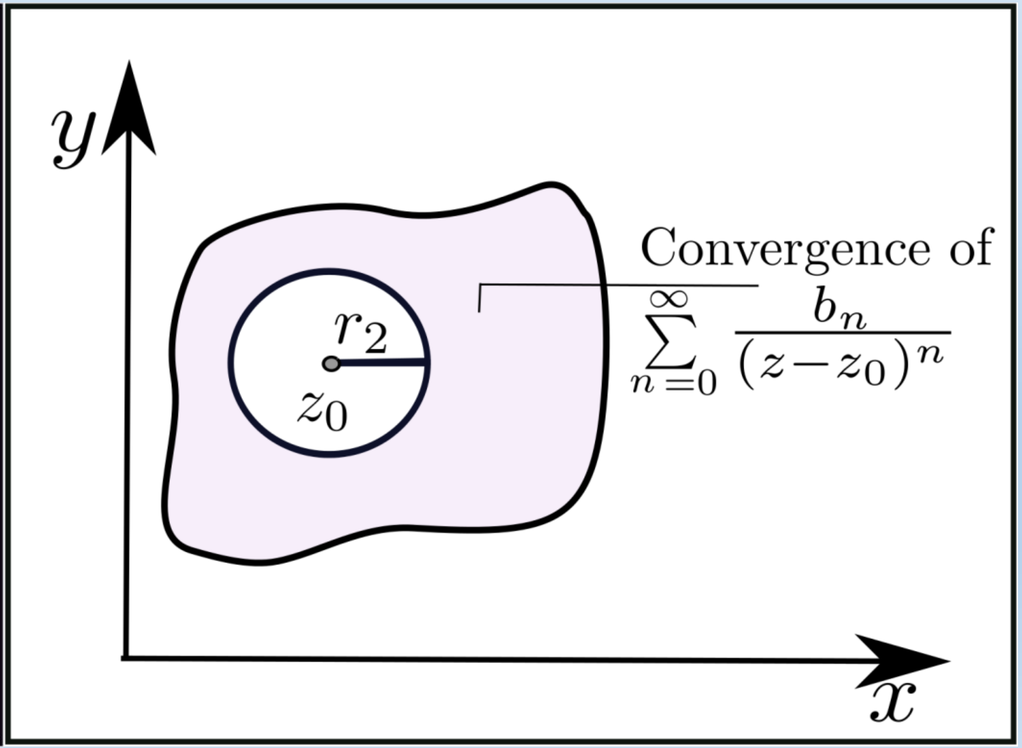
iii) If power series $\sum_{n=0}^{\infty} c_n (z - z_0)^n$ converges for $z_1 \neq z_0$ ,

then it converges absolutely for every value $z$ such that $|z - z_0| < |z_1 - z_0|$.

Similarly, if power series $\sum_{n=0}^{\infty} \frac{b_n}{(z - z_0)^n}$ converge for $z_2 \neq z_0$ then it converges absolutely

for every z such that $|z - z_0| > |z_2 - z_0|$



## Digression –Power series

iv) Convergence of $S=\sum_{n=0}^{\infty} c_n(z-z_0)^n$ means that for partial sums $S_N=\sum_{n=0}^{N} c_n(z-z_0)^n$ and **ε<0**

exists an integer $N_\varepsilon$ such that $|S-S_N|<\varepsilon,\quad N>N_\varepsilon$ .

If $N_\varepsilon$ is the same for all ***z*** inside the circle of convergence, than connvergence is uniform.

v) The power series $S=\sum_{n=0}^{\infty} c_n(z-z_0)^n$ is uniformply convergent for all points within its circle of convergence and **represents a function that is analytic there.**

If the power series $\sum_{n=0}^{\infty} \frac{b_n}{(z-z_0)^n}$ converge in an anulus $r_2<|z-z_0|<r_1$ then it is uniformly convergent and represents analytic function there.

vi) Uniformly convergent power series can be :

a) differentiated term by term within the circle of convergence: $\frac{dS(z)}{dz}=\sum_{n=1}^{\infty} n\cdot c_n(z-z_0)^n$

b)Integrated term by term along any curve which lie entirely inside its circle of convergence:

$$\int_C S(z)dz=\sum_{n=0}^{\infty} c_n\int_C (z-z_0)^n dz$$

# Taylor and Laurent series.

**Taylor's Tm**. Let ***f*** be analytic function in interior of the circle ***C*** centered at $z_0$ and having radius $r_0$ . Then at each point ***z*** inside ***C:***

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}\cdot(z-z_0)^n$$

The region of convergence is determined by $|z-z_0|<r_0$ where $r_0$ is the distance from $z_0$ to nearest singularity of the function ***f(z).***

# Taylor and Laurent series

Taylor's expansion requires that function ***f(z)*** has no singularities inside the circle of convergence. In many occation there may exist singularity in a region of interest.
In this case function may be given by a so called Laurent expansion.

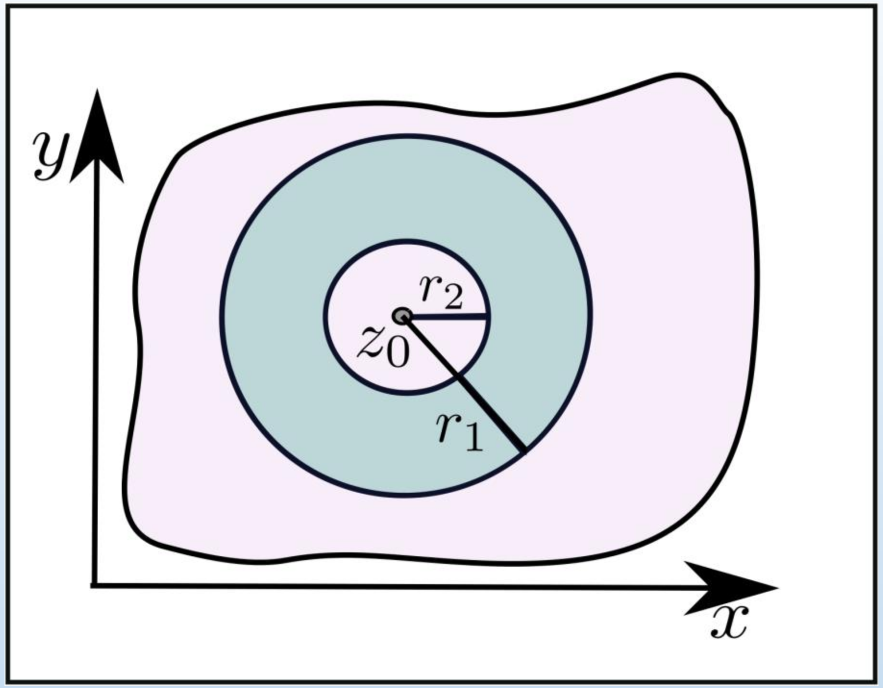
**Tm.** Let $C_1$ and $C_2$ be circles in the ***z*** plane centered at $z_0$ with radii $r_2<r_1$ .Let ***f(z)*** be analytic on $C_1$ and $C_2$ and in an annular region ***R*** between them.

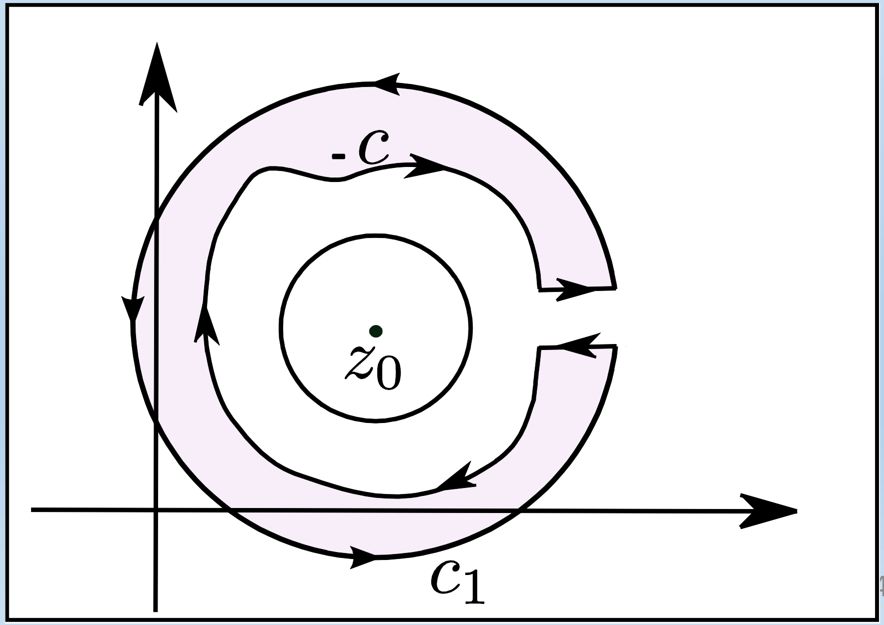
Then for each $z \in R$ , ***f(z)*** is given by:

$$f(z)=\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$

$$c_n = \int_C \frac{f(\xi)\,d\xi}{(\xi - z_0)^n}$$

***C*** is any positively oriented contour within ***R*.**



i) Laurent series converges for $r_2 < |z - z_0| < r_1$ .

ii) If $r_2 = 0, \quad \Rightarrow c_n = 0, n = -1, -2, \ldots$

In that case, Laurent series recover Taylor series (as it should be).

iii) Part with nonnegative powers of ***(z-z₀)***

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n$$

is called analytic or regular part of Laurent series.

iV) Part which consists of inverse powers of $(z-z_0)$

$$\sum_{n=-\infty}^{-1} c_n (z - z_0)^n = \ldots \frac{c_{-n}}{(z - z_0)^n} + \ldots + \frac{c_{-1}}{(z - z_0)}$$

is called principal part.

Complex number, coefficient $c_{-1} = \frac{1}{2\pi \cdot i} \int_C f(z) dz$

is called residue and is denoted by

$$c_{-1} = Res[f(z_0)]$$

Tm. ***Uniqueness of the Laurent expansion***

If the series $\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$ converges to analytic function ***f(z)*** in some annular region about $z_0$, than it is the unique Laurent series expansion in this region.

# Singularities of complex functions

**Def.** Let ***f(z)*** be a complex valued function.
Singular point $z_0$ is a point at which ***f(z)*** is not analytic.
If there is an neighborhood

$$0 < |z - z_0| < r$$

where ***f*** is analytic except $z_0$, then $z_0$ is isolated singular point.

Isolated singular point

Classification of singularities of analytic function ***f*** is possible by examination of its Laurent expansion.
There are several kinds of singularities:
i) Removable
ii) Poles
iii) Essential
iv) Branch points

Classification of singularities

## Removable singularities

A point $z=z_0$ is called removable singularity if function ***f(z)*** is not defined at $z=z_0$ but $\lim_{z\to z_0} f(z)$ exists.

Examples:

$$f(z)=\frac{\sin z}{z}, \qquad \lim_{z\to 0}\frac{\sin z}{z}=1$$

$$f(z)=\frac{e^z-1-z}{z^2}, \qquad \lim_{z\to 0}\frac{e^z-1-z}{z^2}=\frac{1}{2}$$

## Poles

Let principal part of Laurent expansion of analytic function ***f(z)*** around $z=z_0$ has only a finite number of terms:

$$c_{-m}\neq 0, \qquad c_{-n}=0,\ n>m$$

Then function ***f(z)*** has a pole of order m.

$$\lim_{z\to 0} f(z)=\pm\infty$$

If m=1, pole is called simple pole.

## Essential singularities

**Def.** If the principal part of Laurent expantion around point $z_0$ has Infinite number of terms, then $z_0$ is essential singular point of function f(z)

In the neighborhood of essential singularity an, otherwise analytic function f(z) may take any value except possibly one.

Example :

Z=0 is essential singularity of

$$f(z) = e^{1/z}$$

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{z^n}$$

$$\lim_{z \to 0} e^{1/z} = \begin{cases} \infty & \text{z approaches 0 along +x} \\ -\infty & \text{z approaches 0 along –x} \\ \text{oscilating} & \text{z approaches zero along } +iy \end{cases}$$

## Branch point

A point $z_0$ is called branch point of a multivalued function ***f(z)*** if the branches of ***f(z)*** interchanged when ***z*** describes a closed path around $z_0$.

Branch point is **not an isolated singular point** because any circle around $z_0$ leads to interchange of branches of multivalued function.

Examples:

$f(z) = \sqrt{z}$ and $f(z) = \ln(z)$ have a branch point at ***z=0***

## Singularities at infinity

By letting ***z=1/w*** in function ***f(z)***, we obtain the function

$F(w) = f(1/w)$

The nature of singularity at ***z=∞*** is defined to be the same as that of ***F(w)*** at ***w=0***

Examples:

Function $f(z) = z^3$ has the pole of order 3 at infinity becuse $F(w) = 1/w^3$ has a pole of order 3 at zero.

Function $f(z) = e^z$ has essential singularity at infinity because function $F(w) = e^{1/z}$ has essential singularity at zero.

# Classification of functions

Laurent expansion of analytic function may serve as definition of two kinds of analytic functions

**Entire**

- Analytic everywhere except at **z=∞**
- Can be represented by Taylor series which has an infinite radius of convergence.

Examples: $e^z, \sin z, \cos z$

**Meromorphic**

- All singularities in a give region ***R*** are isolated poles and removable singularities - By definition, meromorphyc functions have no essential singularities

**Rational**

$$f(z) = \frac{P_n(z)}{Q_m(z)}$$

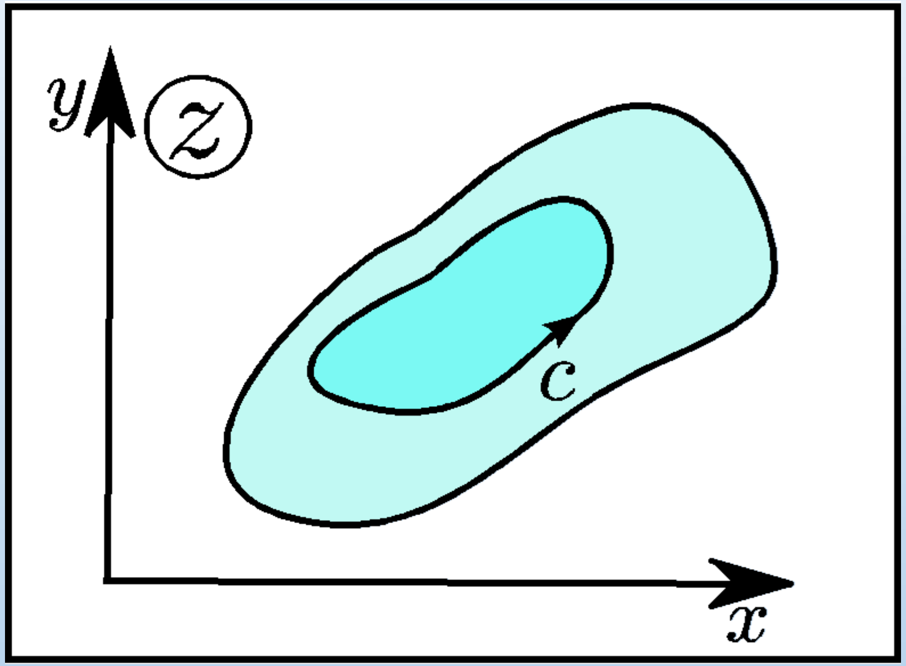
where ***P*** and ***Q*** are polynomials .
Meromorphic in entire complex plane.

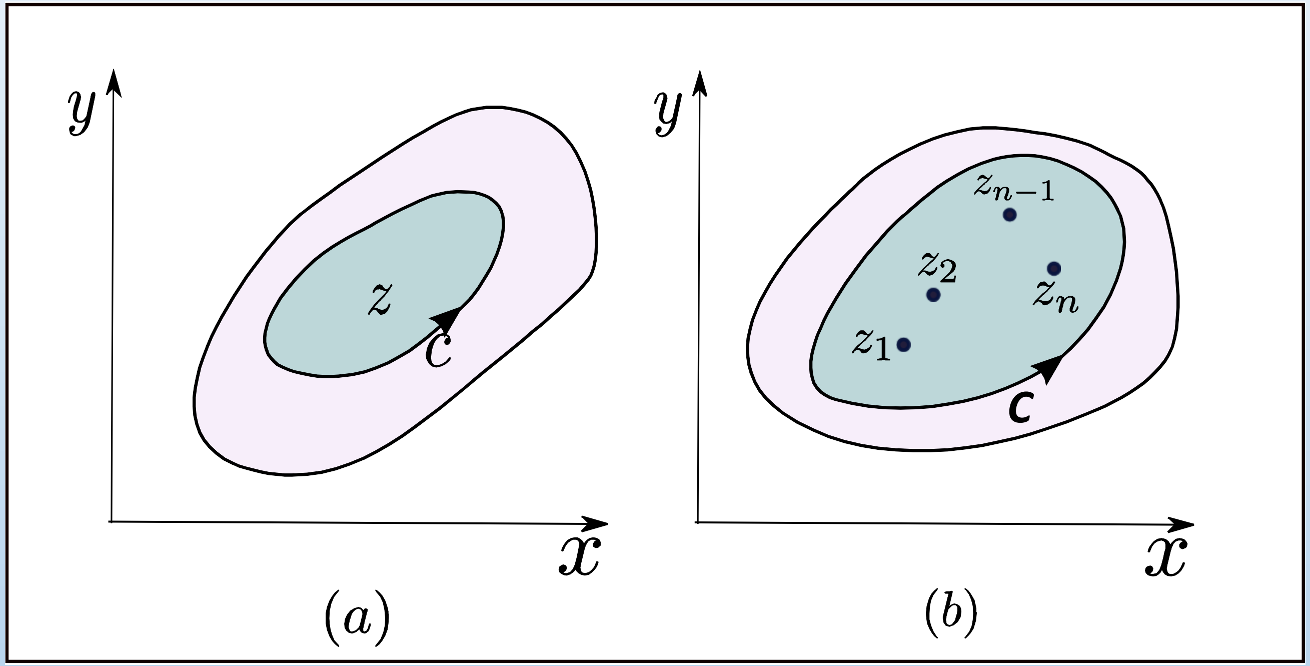
If function ***f(z)*** is meromorphyc in entire complex plane , than ***f(z)*** is rational function

Example: $f(z) = \frac{z}{(z-1)\cdot(z+3)^2}$ analytic everywhere except simple pole at z=1 and a second order pole at z=-3

# The Cauchy's residue theorem

The residue theorem has the same significance for meromorphic functions in range ***R*** as Cauchy's formula for functions analytic in R.




$$\int_C f(z)dz = 0$$

$$f(z) = \int_C \frac{f(\xi)d\xi}{\xi - z}$$

Cauchy's residue theorem for function having poles inside contour ***C***

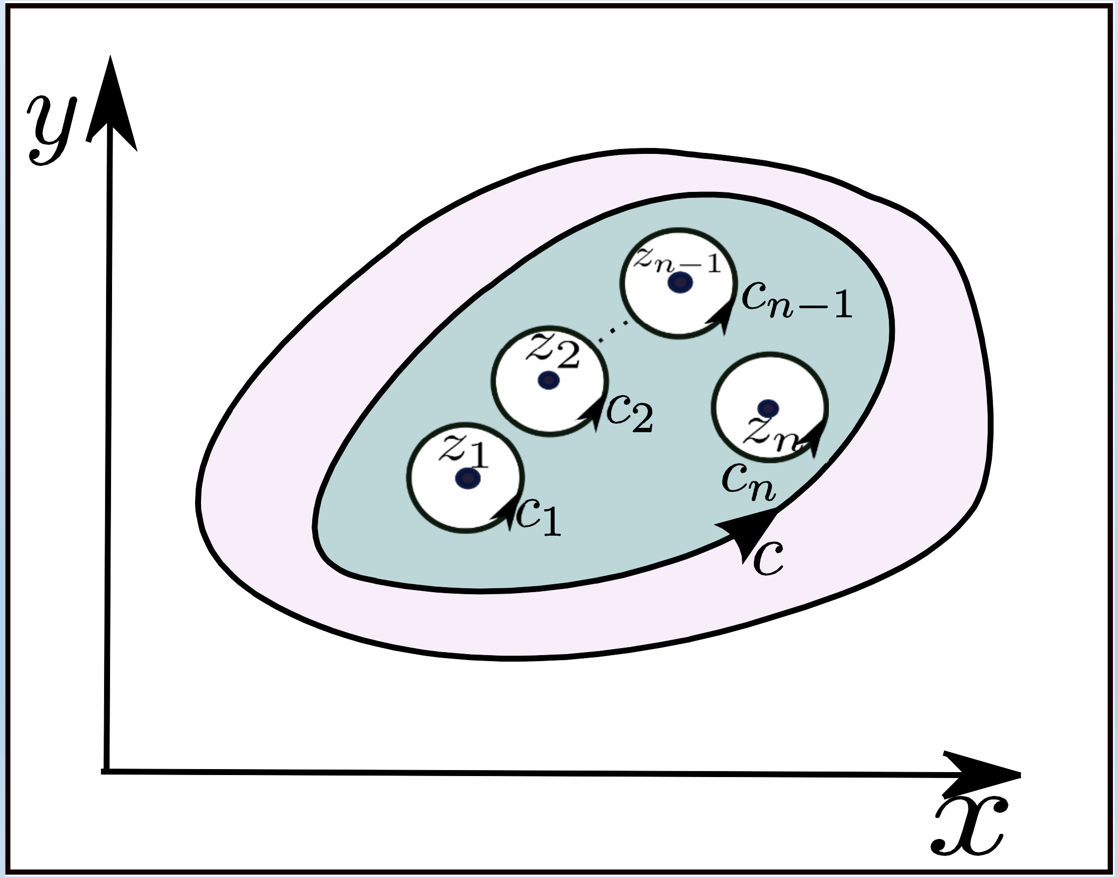
**Analytic inside contour C**

# Cauchy's residue theorem

Let ***f(z)*** be analytic function inside and on the positively oriented contour ***C*** except for a finite number of poles at points $z_1, \ldots, z_n$ .

Then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}\left[f(z_k)\right], \quad \mathrm{Res}\left[f(z_k)\right] = \frac{1}{2\pi i}\int_{C_k} f(z)\,dz$$



# Analytic continuation

Motivation: It is often tha case that analytic function ***f(z)*** is given in a limited region ***R.***
We may ask the question:
*Is it possible to extend the function beyond* ***R***
***Answer: Yes, under certain conditions.***

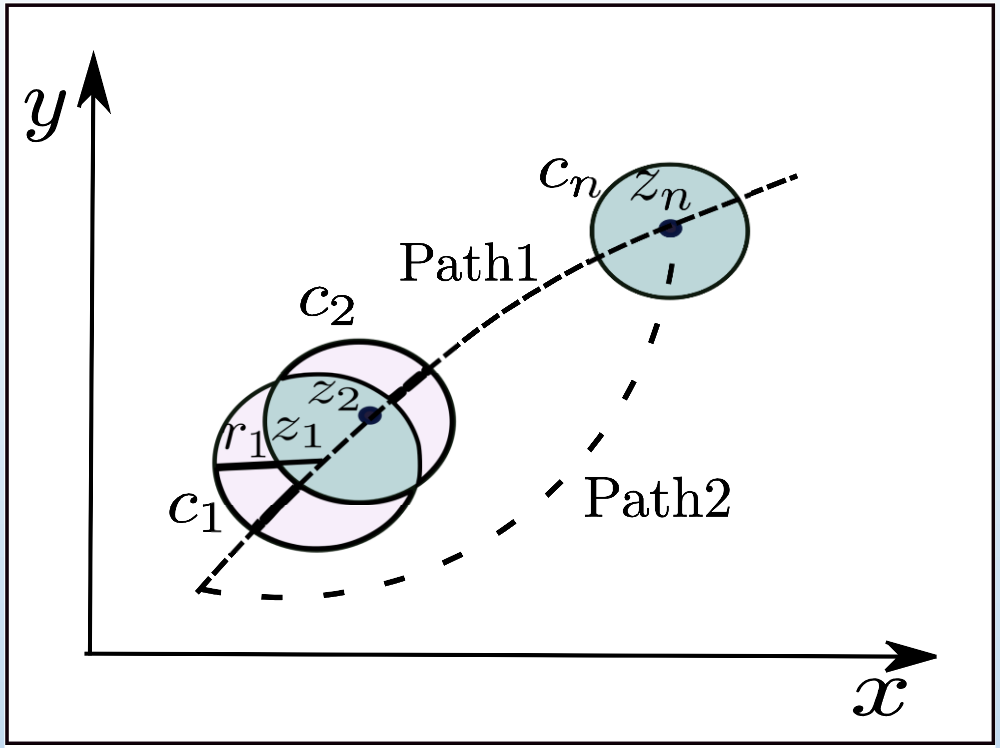
Suppose we do not know precise form of the analytic function inside the circle of convergence $C_1$ with radius of convergence $r_1$ . ***f(z)*** is represented by a Taylor expansion:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

1. Calculate ***f(z)*** and all its derivatives in point $z_2$ inside $C_2$ and arrive to expression:

$$f(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$$

having circle of convergence $C_2$ and radius of convergenc $r_2$ ( no singularity on $C_1$ inside $C_2$ ).
2. Repeat this procedure until arrive to point $z_n$



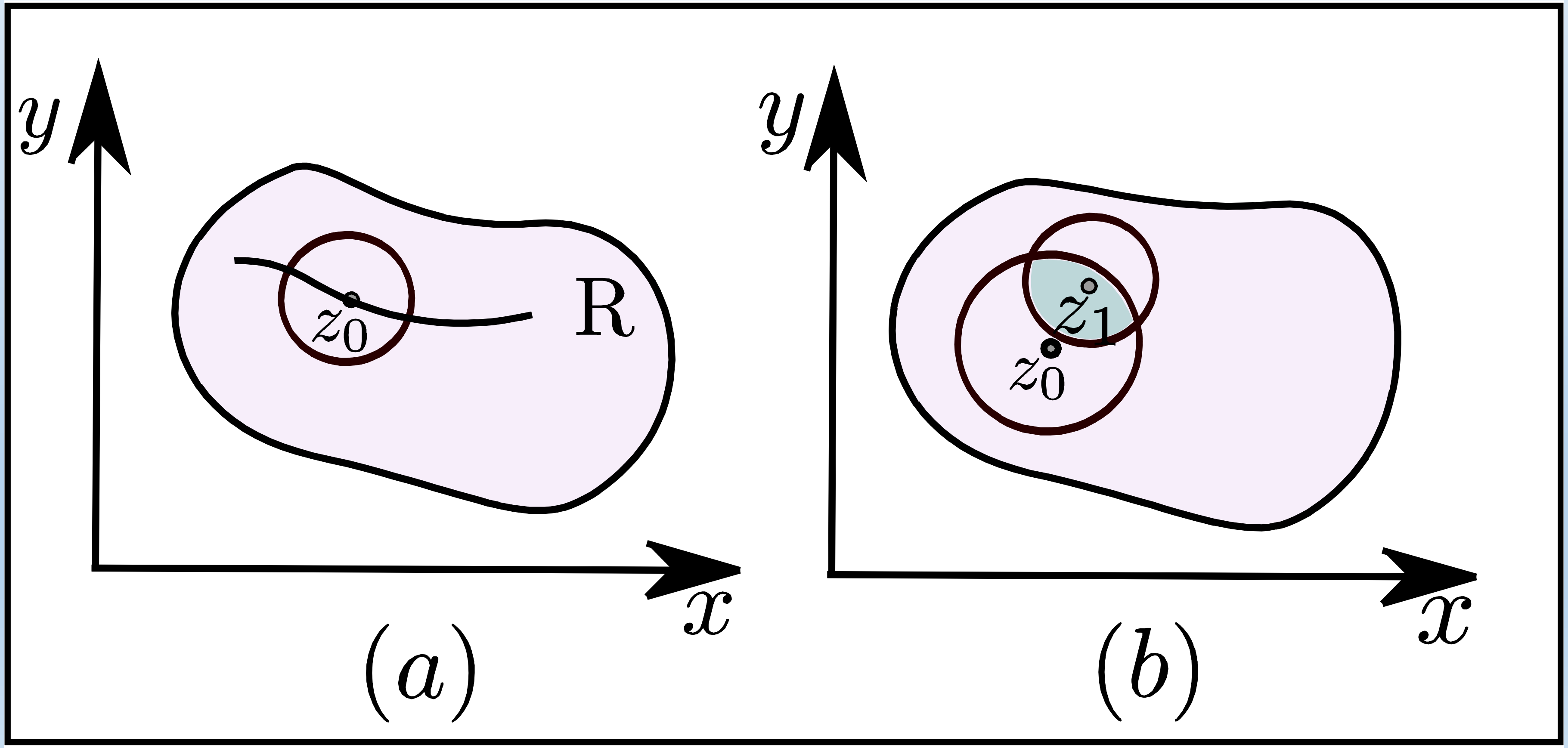

We say that ***f(z)*** is extended analytically beyond $C_1$. The procedure is called **analytic continuation**.

**Q:**
**Is analytic continuation unique?**
**Shall we obtain the same result using Pat2 instead of the Path1?**

## Fundamental theorems

**Tm1.** If the function ***f*** is analytic in a region ***R*** and vanisches in neighborhood of $z_0 \in \boldsymbol{R}$ or for a segment of curve in ***R.***
Then it vanishes identically in this region

Let ***f1*** and ***f2*** are analytic in ***R***. If ***f1=f2*** in a neighborhood of a point ***z***, or for a segment of curve in ***R***,
then ***f1=f2*** in ***R.***



**Tm2**. Let $f_1$ and $f_2$ be analytic in regions $R_1$ and $R_2$ respectively.
Suppose $f_1$ and $f_2$ have different functional forms in their respective regions of analyticity . If there is an overlap between $R_1$ and $R_2$ and if $f_1=f_2$ within that overlap,
then $f_2$ is unique analytic continuation of $f_1$ in $R_2$ and vice versa.
We may regard $f_1$ and $f_2$ as a single function $f$ in $R = R_1 \cup R_2$ such that

$$f(z) = \begin{cases} f_1(z) & z \in R_1 \\ f_2(z) & z \in R_2 \end{cases}$$

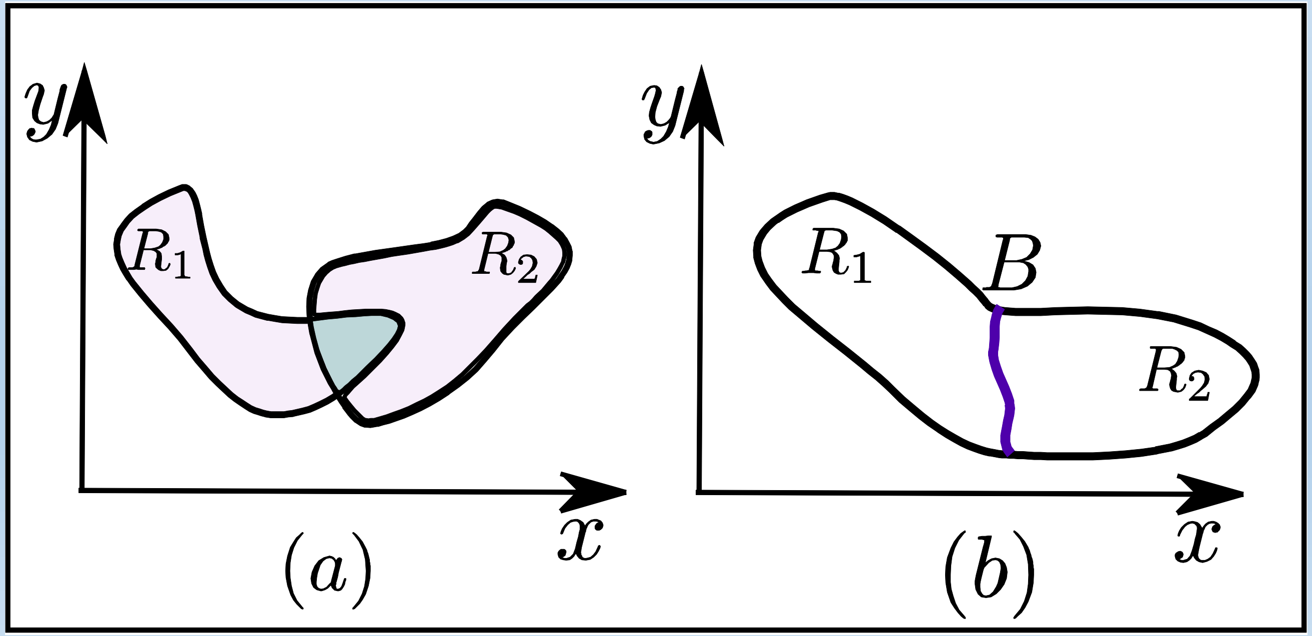
This theorem holds even if regions $R_1$ and $R_2$ have a common boundary $B$ and $f_1=f_2$ on it.
Then:

$$f(z) = \begin{cases} f_1(z) & z \in R_1 \cup B \\ f_2(z) & z \in R_2 \cup B \end{cases}$$

Which is analytic in $R_1 \cup R_2 \cup B$ .

**Tm3.** Continuation of analytic function $f$ from $z_0$ to $z_1$ along two paths is unique.

If two different values of function at $z_1$ are obtained, than $f(z)$ must have a branch point between these paths



**Example**. Let's consider function :

$$f_1(z) = \sum_{n=0}^{\infty} z^n \text{ which is analytic for } |z| < 1.$$

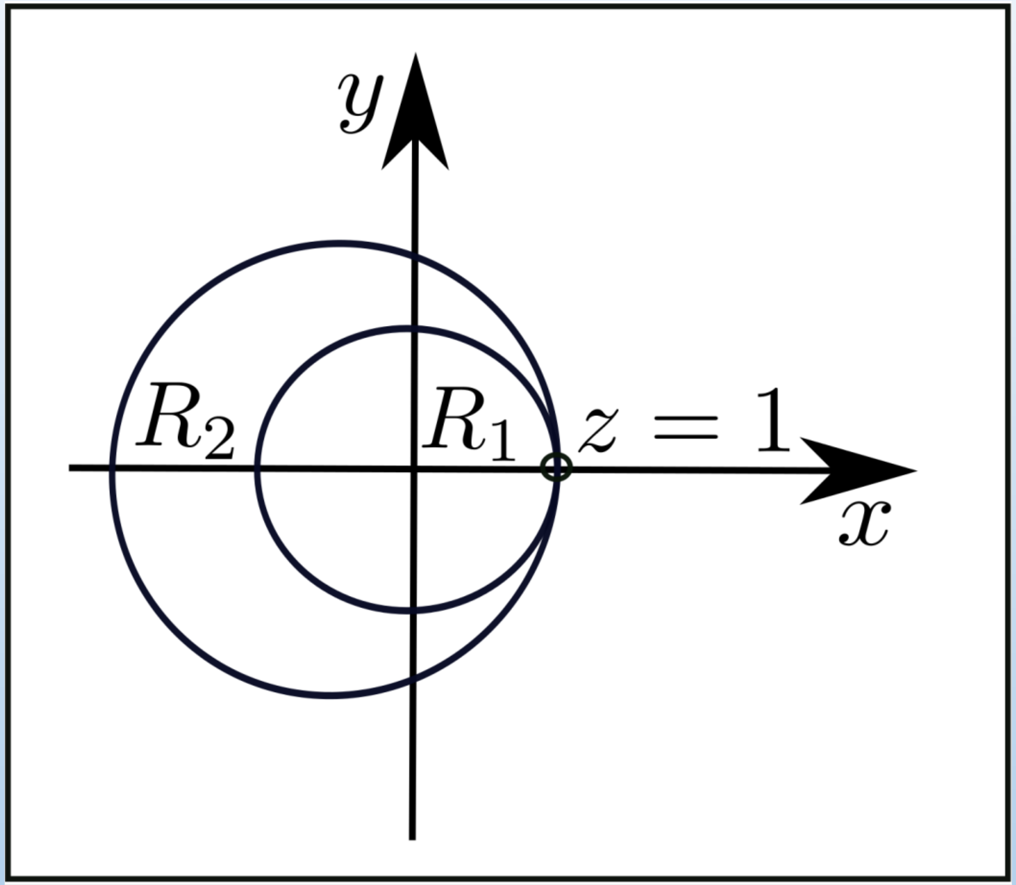
$$f_1(z) = \frac{1}{1-z} \text{ for } |z| < 1 \text{ and not defined for } |z| > 1$$

$$f_2(z) = \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^{n+1} \left(z + \frac{2}{3}\right)^n = \frac{3}{5} \sum_{n=0}^{\infty} \left[\frac{3}{5}\left(z + \frac{2}{3}\right)\right]^n \text{ converges for } \left|z + \frac{2}{3}\right| < \frac{5}{3}$$

Its sum is :

$$f_2(z) = \frac{3}{5} \frac{1}{1 - \frac{3}{5}\left(z - \frac{2}{3}\right)} = \frac{1}{1-z}$$

Since the power series $f_1$ and $f_2$ represent tha same function in the Common region, they are analytic continuation of each other.
$f_1$ is continued analytically into larger circle.



# Schwarz reflection principle

**Tm**. Let ***f(z)*** be a function that is analytic in a region ***R*** that has a segment of real axis as a part of its boundary B. If ***f(z)*** is real whenever is ***z*** real, than analytic continuation ***g(z)*** of function ***f(z***) into ***R**** ( the mirror image of ***R*** with respect to the real axis x) exists and is given by:

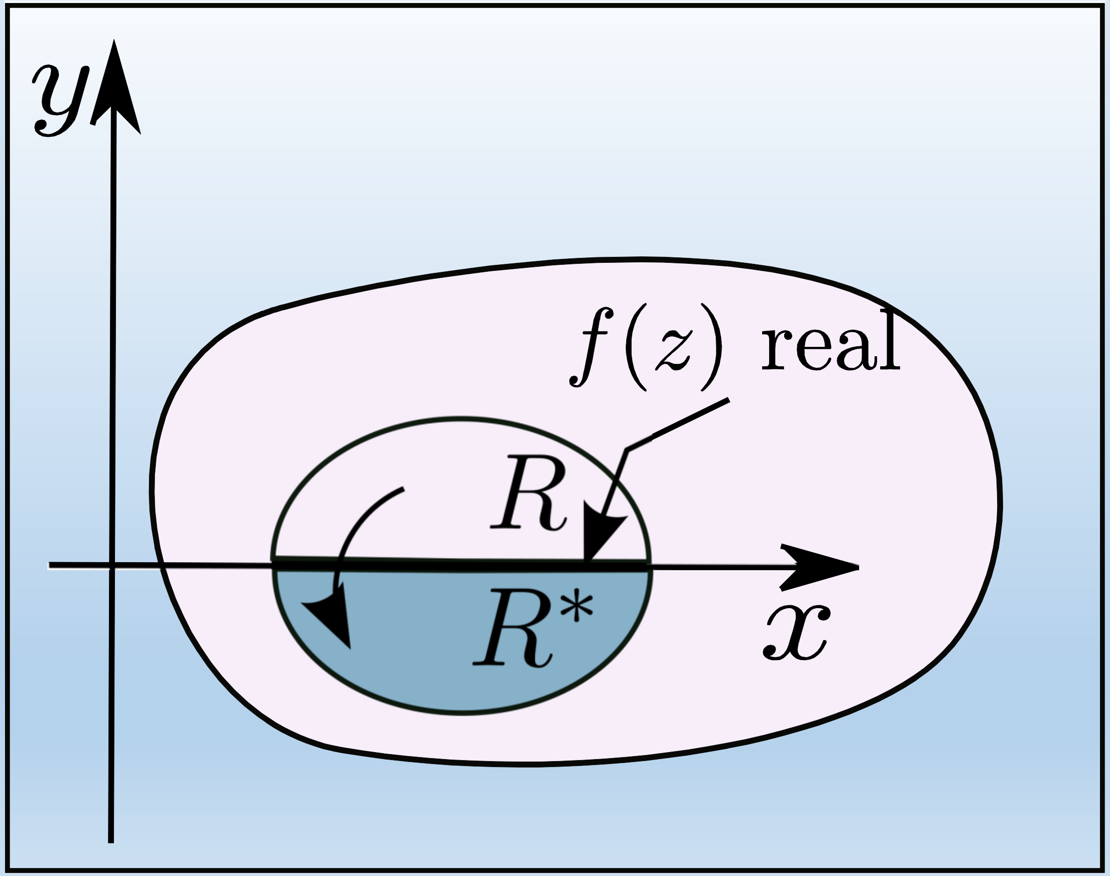
$$g(z) = f^*(z^*) \qquad z \in R^*$$

Since ***f(x,0)=g(x(0)*** on the part of real axis, there exist an analytic function ***h(z)*** such that:

$$h(z) = \begin{cases} f(z) & z \in R \\ g(z) & z \in R^* \end{cases}$$

$$h(z^*) = g(z^*) = f^*(z) = h^*(z)$$

$$f^*(z) = f(z^*)$$



## *Dispersion relations*

Consider an analytic function having a cut along $+\,x$ axis for $x_0 \le x < \infty$ as shown in a figure.

Fro Cauchy's integralformula :

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)\,d\xi}{\xi - z} = \frac{1}{2\pi i}\left\{ \int_{x_0+i\varepsilon}^{R} \frac{f(x+i\varepsilon)\,dx}{x+i\varepsilon-z} + \int_{C_R} + \int_{R}^{x_0-i\varepsilon} \frac{f(x-i\varepsilon)\,dx}{x-i\varepsilon-z} + \int_{C_\rho} \right\}$$

Let's suppose that the integrals around $C_R$ and $C_\rho$ vanish as $R \to \infty$ and $\rho \to 0$.

Than :

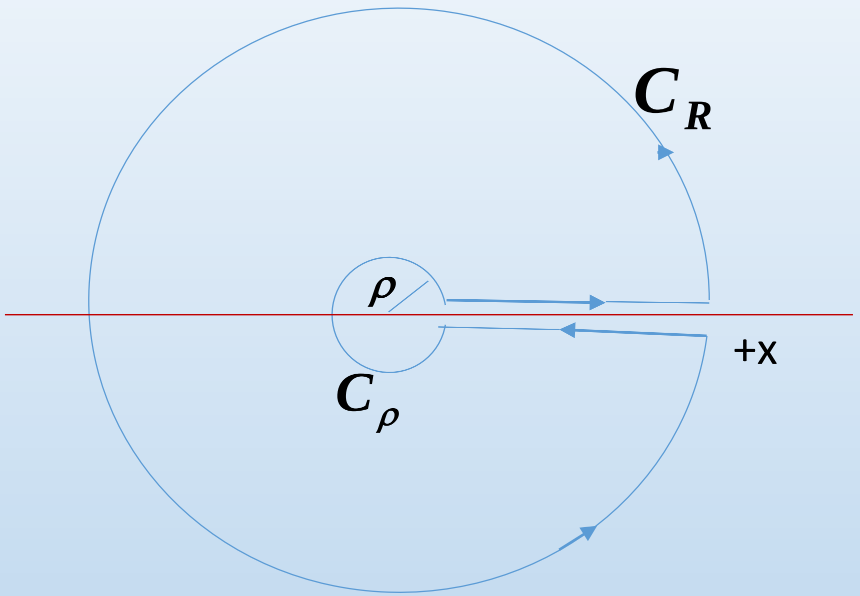
$$f(z) = \frac{1}{2\pi i}\left\{ \int_{x_0}^{\infty} \frac{f(x+i\varepsilon)\cdot dx}{x-z} + \int_{\infty}^{x_0} \frac{f(x-i\varepsilon)\cdot dx}{x-z} \right\}$$

$$f(z) = \frac{1}{2\pi i}\left\{ \int_{x_0}^{\infty} \frac{f(x+i\varepsilon) - f(x-i\varepsilon)}{x-z}\,dx \right\}$$

$$f(z) = \frac{1}{\pi}\int_{x_0}^{\infty} \frac{Im\; f(x+i\varepsilon)}{x-z}\,dx$$

Dispersion relations

Dispersion relation expresses the value of an analytic function at any point of the complex plane in therms of an integral of the imaginary part of the function on the upper edge of the cut.



If $f(z)$ is a function of physical interes, we may write DR going to the limit $z = x' + i\varepsilon$

$$f(x' + i\varepsilon) = \frac{1}{\pi} \int_{x_0}^{\infty} \frac{Im\ f(x + i\varepsilon)}{x - x' - i\varepsilon} dx$$

$$= \frac{1}{\pi} P \int_{x_0}^{\infty} \frac{Im\ f(x + i\varepsilon)}{x - x'} dx + \frac{1}{\pi} i\pi\ Im\ f(x')$$

$$Re\ f(x') = \frac{1}{\pi} P \int_{x_0}^{\infty} \frac{Im\ f(x)}{x - x'} dx,$$

where $P$ stands for 'Principal value integral' (Hauptwertitegral).

We used a very useful simbolic equation :

$$\lim_{\varepsilon \to 0} \frac{1}{x - x' \pm i\varepsilon} = P \frac{1}{x - x'} \mp i\pi\, \delta(x - x').$$

If PVI does not converge, or $f(x)$ does not fall to zero fast enough for numerical calculation, one may use so called once subtracted DR in which is introduced an extra factor of x' in denominator. Once subtracted DR may be obtained using as a function $f(z)/(z\text{-}z_s)$ instead of $f(z)$:

$$f(x') = f(x_s) + \frac{x' - x_s}{\pi} P\int_{x_0}^{\infty} \frac{Im\, f(x)}{(x - x')(x' - x_s)} dx.$$

Formally, once subtracted DR may be obtained simply by calculating $f(x_s)$ using DR and subtract it from DR for $f(x)$:

$$Re\, f(x_s) = \frac{1}{\pi} P\int_{x_0}^{\infty} \frac{Im\, f(x)}{x - x_s} dx,$$

$$Re\, f(x') - Re\, f(x_s) = \frac{1}{\pi} P\int_{x_0}^{\infty} Im\, f(x)\left(\frac{1}{x - x'} - \frac{1}{x - x_s}\right) dx$$

$$Re\, f(x') = Re\, f(x_s) + \frac{x' - x_s}{\pi} P\int_{x_0}^{\infty} \frac{Im\, f(x')}{(x - x')(x - x_s)} dx$$

Once subtracted DR