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Linearly reducible Feynman integrals

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Plan of the talk

1. The hyperlogarithm method of integration [16, 15]
2. Linearly reducible graphs [16, 17, 14]
3. Polynomial reduction [7]
4. Recursions and forest polynomials [16]

Feynman integrals in Schwinger parameters

Scalar propagators $(p_e^2 + m_e^2)^{-a_e}$, $\mathrm{sdd} = \sum_e a_e - D/2 \cdot \mathrm{loops}(G)$:

$$\Phi(G) = \frac{\Gamma(\mathrm{sdd})}{\prod_e \Gamma(a_e)} \int_0^\infty \psi^{\mathrm{sdd}-D/2} \cdot \varphi^{-\mathrm{sdd}} \cdot \prod_{e \in E} \alpha_e^{a_e - 1} \mathrm{d}\alpha_e \cdot \delta(1 - \alpha_N)$$

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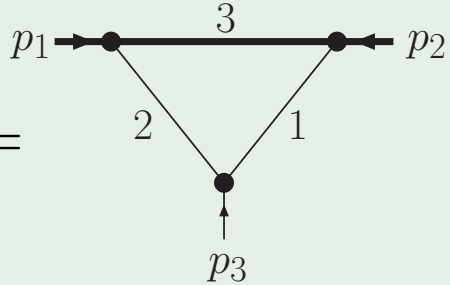
Graph polynomials:

$$\psi = \mathcal{U} = \sum_T \prod_{e \notin T} \alpha_e \qquad \varphi = \mathcal{F} = \sum_{F = T_1 \dot\cup T_2} q^2(T_1) \prod_{e \notin F} \alpha_e + \psi \sum_e m_e^2 \alpha_e$$

Example (arbitrary ε)

$$\Phi(G) = \int_0^\infty \frac{\Gamma(1+\varepsilon)\, \delta(1-\alpha_N)\, \mathrm{d}\alpha_1\, \mathrm{d}\alpha_2\, \mathrm{d}\alpha_3}{(\alpha_1+\alpha_2+\alpha_3)^{1-2\varepsilon} \left[m^2(\alpha_1+\alpha_2+\alpha_3) + p_1^2\alpha_2 + p_2^2\alpha_1\right]^{1+\varepsilon} \alpha_3^{1+\varepsilon}}$$

$G =$ 

$D = 4 - 2\varepsilon \qquad a_e = 1 \qquad \mathrm{sdd} = 1 + \varepsilon$

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Example (expanded in $\varepsilon \to 0$)

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$$= \Gamma(1+\varepsilon) \sum_{n=0}^\infty \frac{\varepsilon^n}{n!} \int_0^\infty \frac{\delta(1-\alpha_N)\, \mathrm{d}\alpha_1\, \mathrm{d}\alpha_2\, \mathrm{d}\alpha_3}{\psi\varphi} \log^n \frac{\psi^2}{\varphi}$$

Hyperlogarithms and multiple polylogarithms

Definition (Poincaré, Lappo-Danilevsky)

Set $G(0;z) := \log(z)$ and for sequences $\sigma_1, \ldots, \sigma_n \in \mathbb{C}$ with $\sigma_n \neq 0$ let

$$G(\sigma_1, \ldots, \sigma_n; z) := \int_0^z \frac{\mathrm{d}z_1}{z_1 - \sigma_1} \int_0^{z_1} \frac{\mathrm{d}z_2}{z_2 - \sigma_2} \cdots \int_0^{z_{n-1}} \frac{\mathrm{d}z_n}{z_n - \sigma_n}.$$

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These iterated integrals are multiple polylogarithms

$$G(\underbrace{0, \ldots, 0, \sigma_r}_{n_r}, \cdots \underbrace{0, \ldots, 0, \sigma_1}_{n_1}; z) = (-1)^r \operatorname{Li}_{n_1, \ldots, n_r}\left(\frac{\sigma_2}{\sigma_1}, \ldots, \frac{\sigma_r}{\sigma_{r-1}}, \frac{z}{\sigma_r}\right),$$

where we have for $|z_r|, |z_{r-1}z_r|, \ldots, |z_1 \cdots z_r| < 1$ that

$$\operatorname{Li}_{n_1, \ldots, n_r}(z_1, \ldots, z_r) = \sum_{0 < k_1 < \cdots < k_r} \frac{z_1^{k_1} \cdots z_r^{k_r}}{k_1^{n_1} \cdots k_r^{n_r}}.$$

Integration with hyperlogarithms following Brown [8]

Applications by Chavez & Duhr [11], Wißbrock [1], Anastasiou et. al. [3, 2]

To compute $\int_0^\infty f \, \mathrm{d}\alpha_e$ where f is a rational linear combination of polylogarithms that depend rationally on α_e:

1. Rewrite f using hyperlogarithms:

$$f = \sum_{w,\sigma,n} \frac{G(w;\alpha_e)}{(\alpha_e - \sigma)^n} \lambda_{w,\sigma,n} \quad \text{with constants } \lambda_{w,\sigma,n} \text{ w.r.t. } \alpha_e.$$

Implemented in the `Maple` code `HyperInt` [15], completely algebraic.
Assumption: Finite integrals! No restriction, see [17, 22]

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2. Construct an antiderivative $\partial_{\alpha_e} F = f$.
3. Evaluate the limits

$$\int_0^\infty f \, \mathrm{d}\alpha_e = \lim_{\alpha_e \to \infty} F(\alpha_e) - \lim_{\alpha_e \to 0} F(\alpha_e).$$

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Linear reducibility

Precondition: For all $n < N$,

$$f_n := \left[\prod_{e=1}^{n-1} \int_0^\infty \mathrm{d}\alpha_e\right] \psi^{\mathrm{sdd}\,-D/2} \varphi^{-\,\mathrm{sdd}} \prod_{e\in E} \alpha_e^{a_e-1}$$

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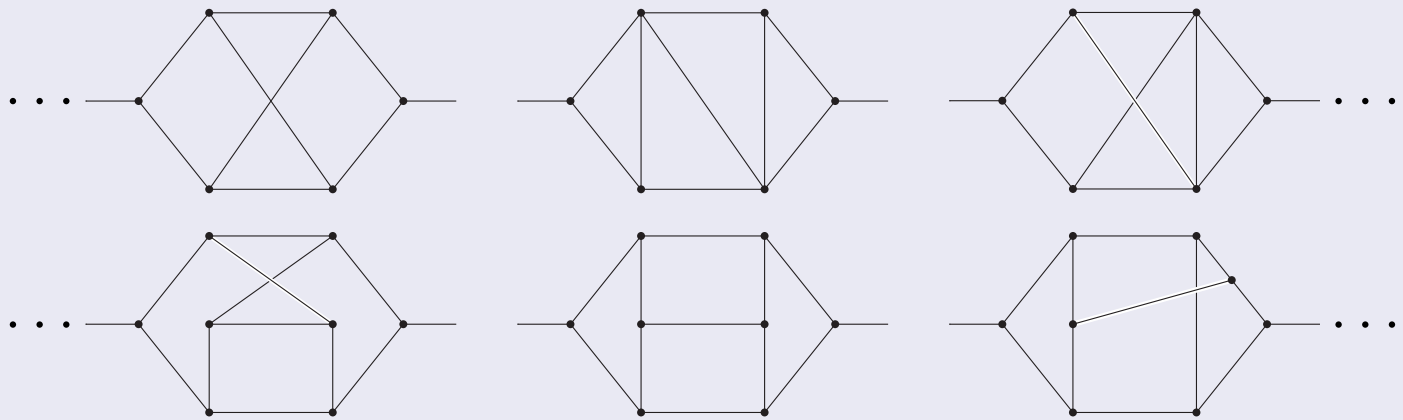
- Combinatorial condition on the polynomials ψ and φ only; independent of ε-order and expansion point $(D, \vec{a})_{\varepsilon=0} \in 2\mathbb{N} \times \mathbb{Z}^N$
- Polynomial reduction algorithms [8, 7] available (e.g. `HyperInt`) to check sufficient criteria for linear reducibility

Are there linearly reducible Feynman graphs?

Massless propagators

Theorem (four loops; [14])

All massless propagators up to four loops are linearly reducible and their ε-expansions give (at worst) alternating Euler sums $\mathrm{Li}_{n_1,\ldots,n_r}(\pm 1,\ldots,\pm 1)$.

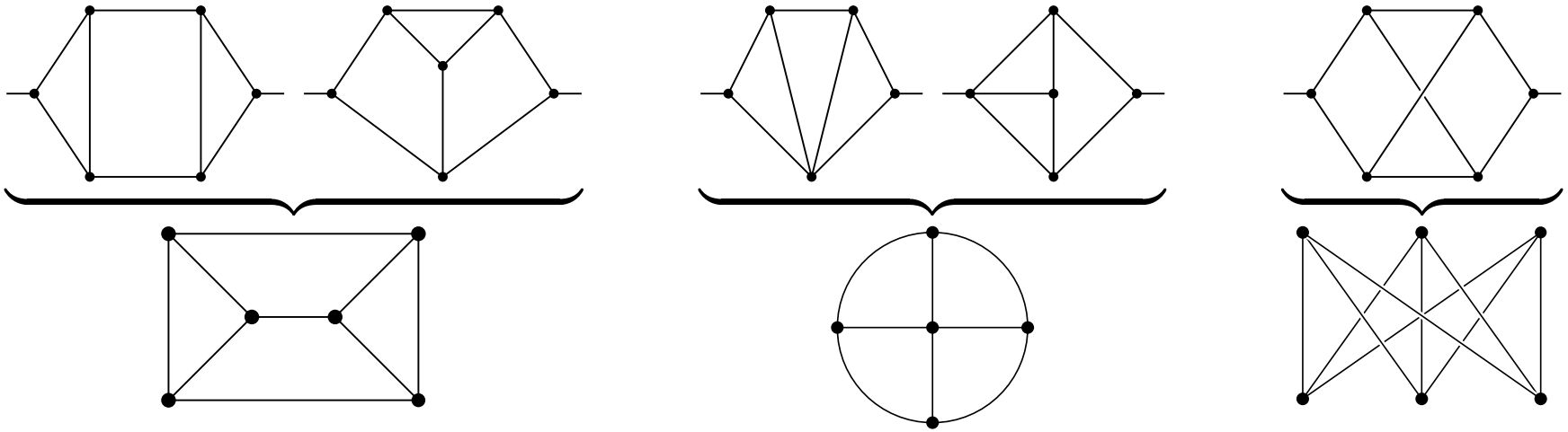


Hence all these graphs can be computed *analytically*

- to any order in ε expanded near arbitrary even dimension $D|_{\varepsilon=0} \in 2\mathbb{N}$,
- with any tensor structures and
- for arbitrary powers $a_e = n_e + \varepsilon\nu_e$ of propagators ($n_e \in \mathbb{Z}, \nu_e \in \mathbb{C}$).

Massless propagators: Glueing

Join external legs of propagator G into edge $e_0 \rightsquigarrow$ vacuum graph $\widehat{G}$:



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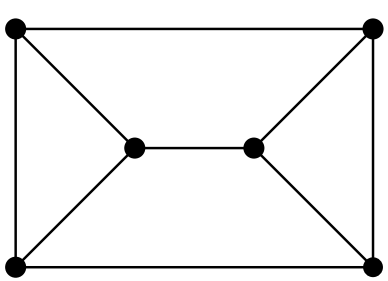
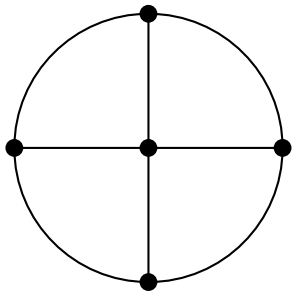
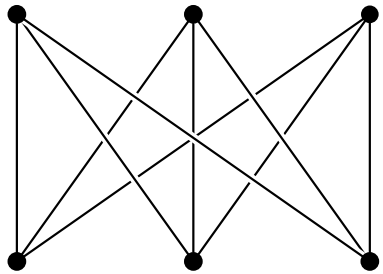

Because $\psi_{\widehat{G}} = \alpha_{e_0}\psi_G + \varphi_G$, linear reducibility (and the Feynman integral) of a massless propagator is equivalent to that of the glued graph.

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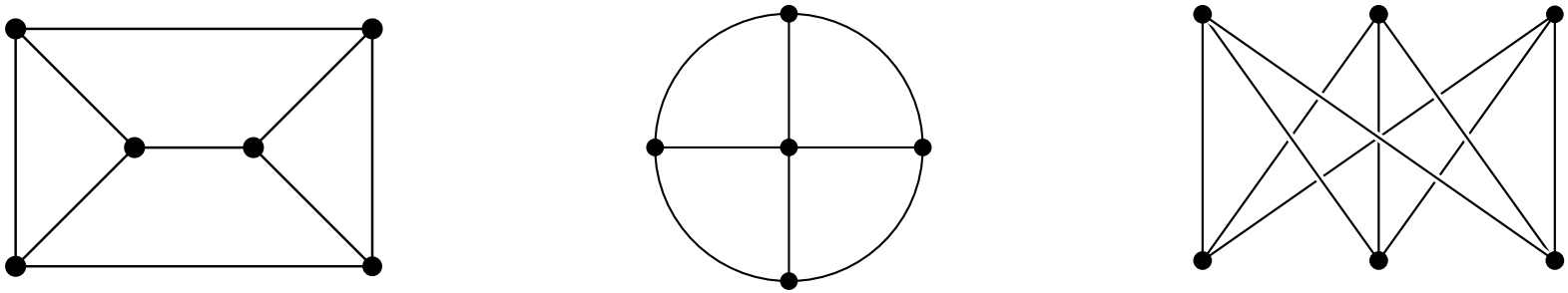

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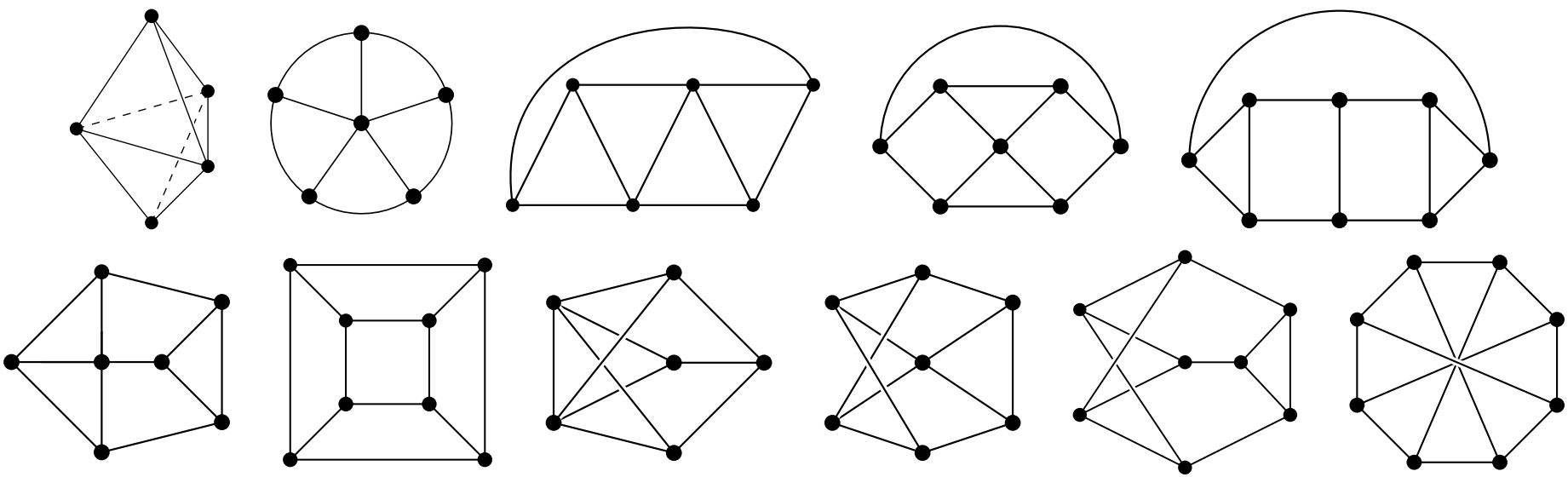
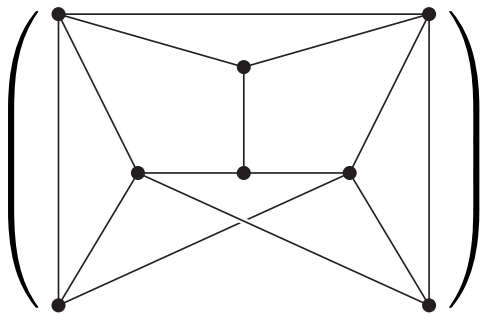
All 3-connected 5-loop vacuum graphs:



ϕ^4-periods up to seven loops

Most primitive log-divergent ϕ^4-periods are linearly reducible, e.g. [6, $P_{7,9}$]:

$$\mathcal{P}\left(\quad\right) = \frac{92943}{160}\zeta_{11} + \frac{3381}{20}\left(\zeta_{3,5,3} - \zeta_{3,5}\zeta_3\right) - \frac{1155}{4}\zeta_3^2\zeta_5$$
$$+ 896\zeta_3\left(\frac{27}{80}\zeta_{3,5} + \frac{45}{64}\zeta_3\zeta_5 - \frac{261}{320}\zeta_8\right)$$



All primitive, $\leq$ 7-loop ϕ^4-periods known exactly; non-polylogs at 8 loops [10, 9].

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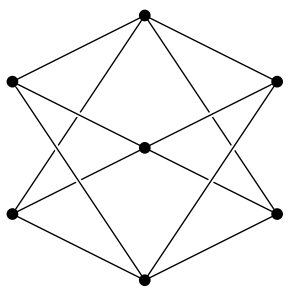
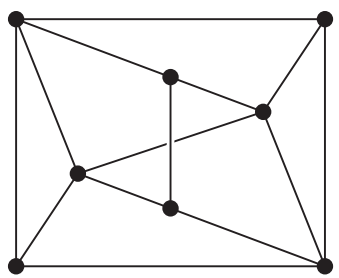
All primitive, $\leq$ 7-loop ϕ^4-periods known exactly; non-polylogs at 8 loops [10, 9].The only non-linearly-reducible ϕ^4-periods at six and seven loops:

$K_{3,4} =$ 

Integrable with *graphical functions*, O. Schnetz [19]. Extremely efficient (graphs up to ten loops).

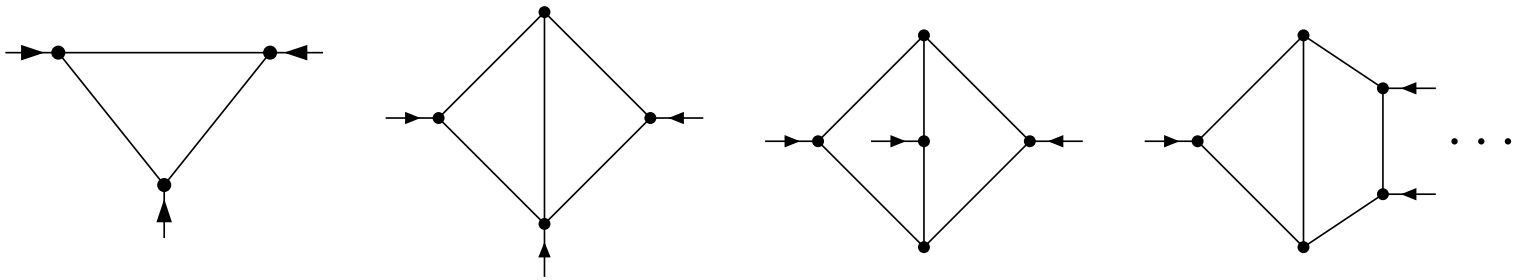
$P_{7,11} =$ 

Linearly reducible after change of variables. Does not give a multiple zeta value!

Linearly reducible graphs: 3-point

Off-shell massless three-point integrals ($m_e = 0$ and $p_1^2, p_2^2, p_3^2 \neq 0$):

- All up to three loops [17]



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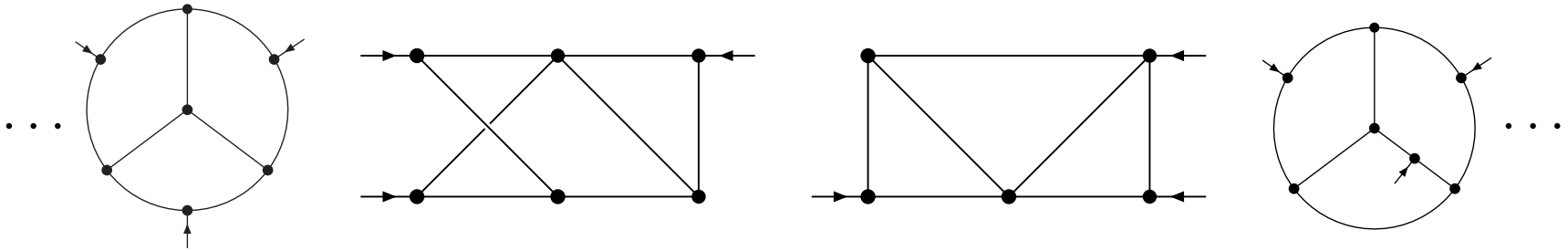

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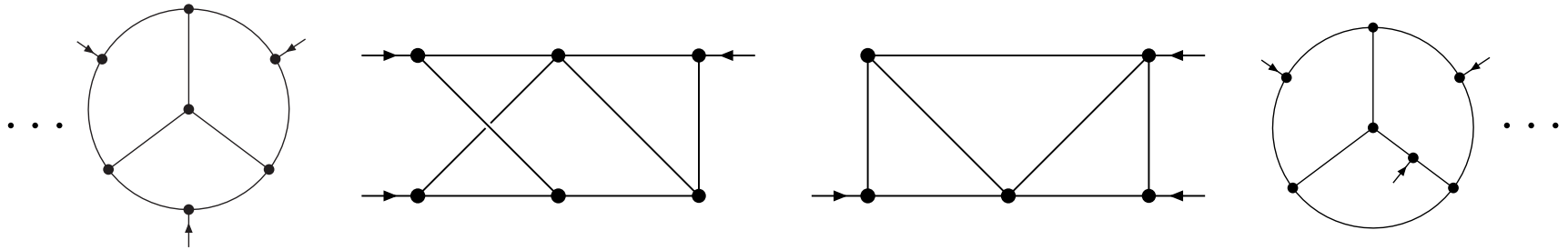
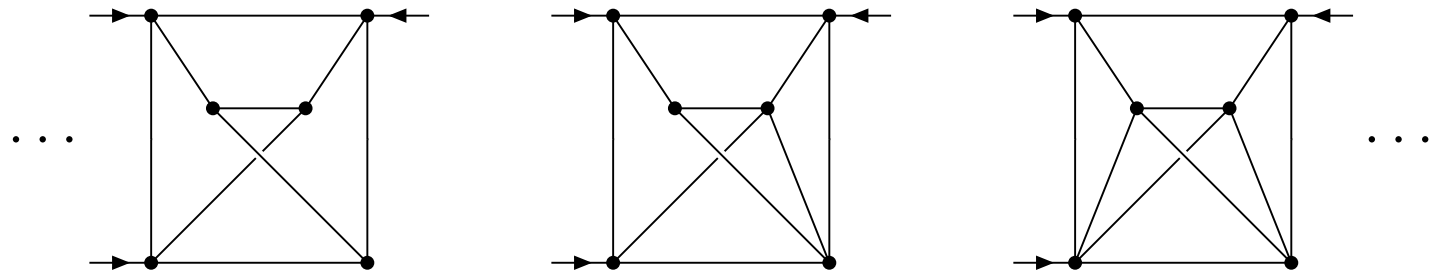


- Some (not all) at four loops and above [17]



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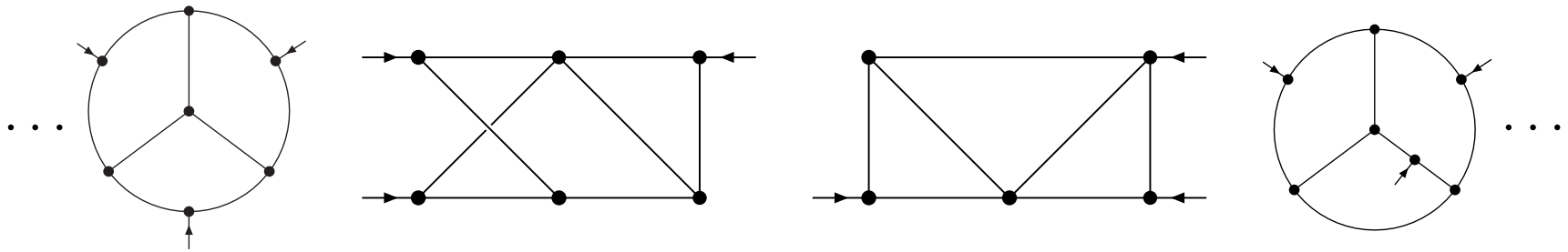

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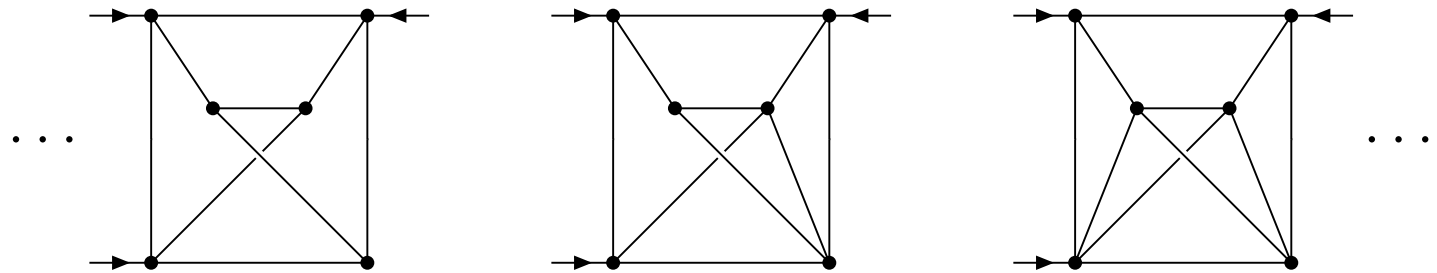
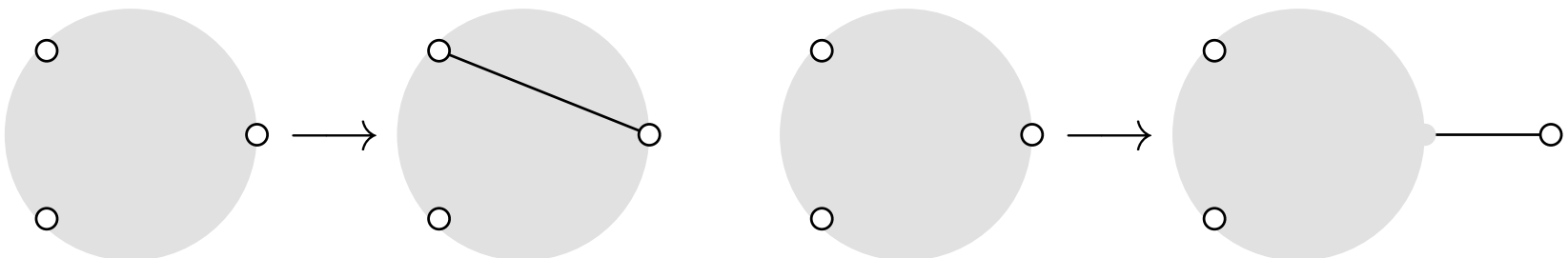


- All with vertex-width three [16]



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Massless on-shell four-point graphs ($m_e = p_1^2 = \cdots = p_4^2 = 0$):

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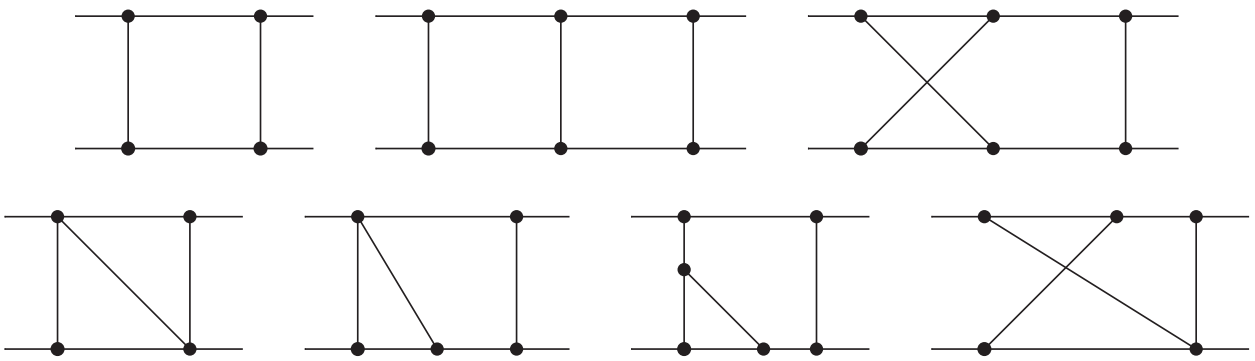

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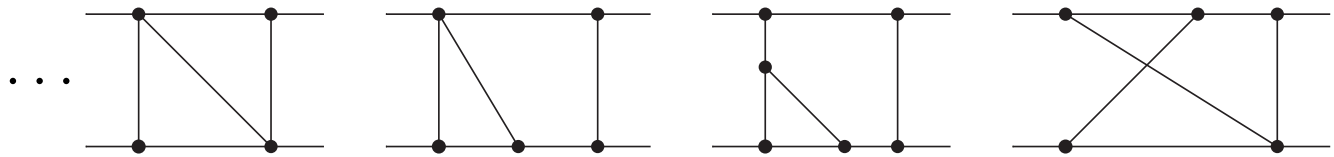
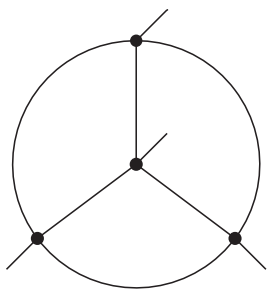


- Counterexamples at three loops, like

$K_4 =$ 

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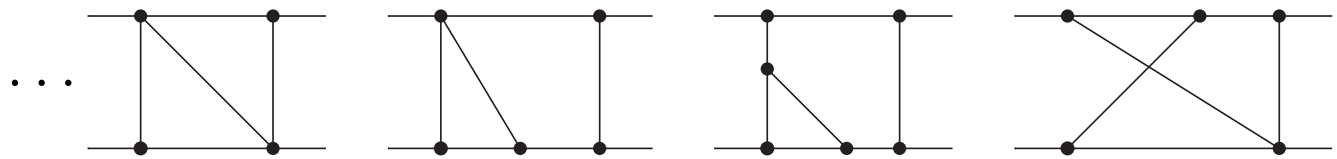
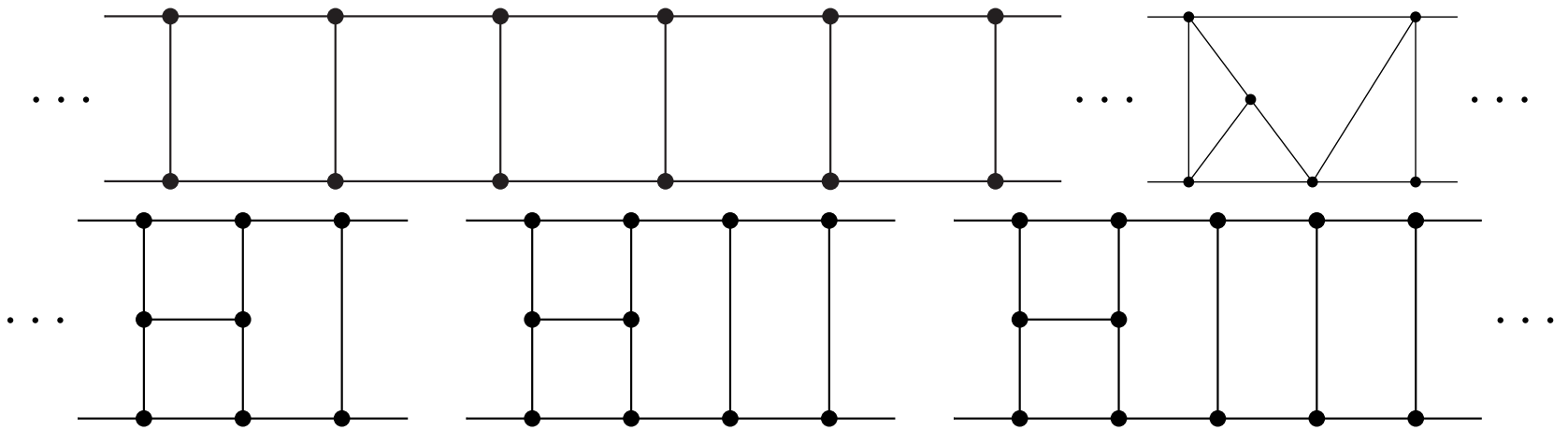


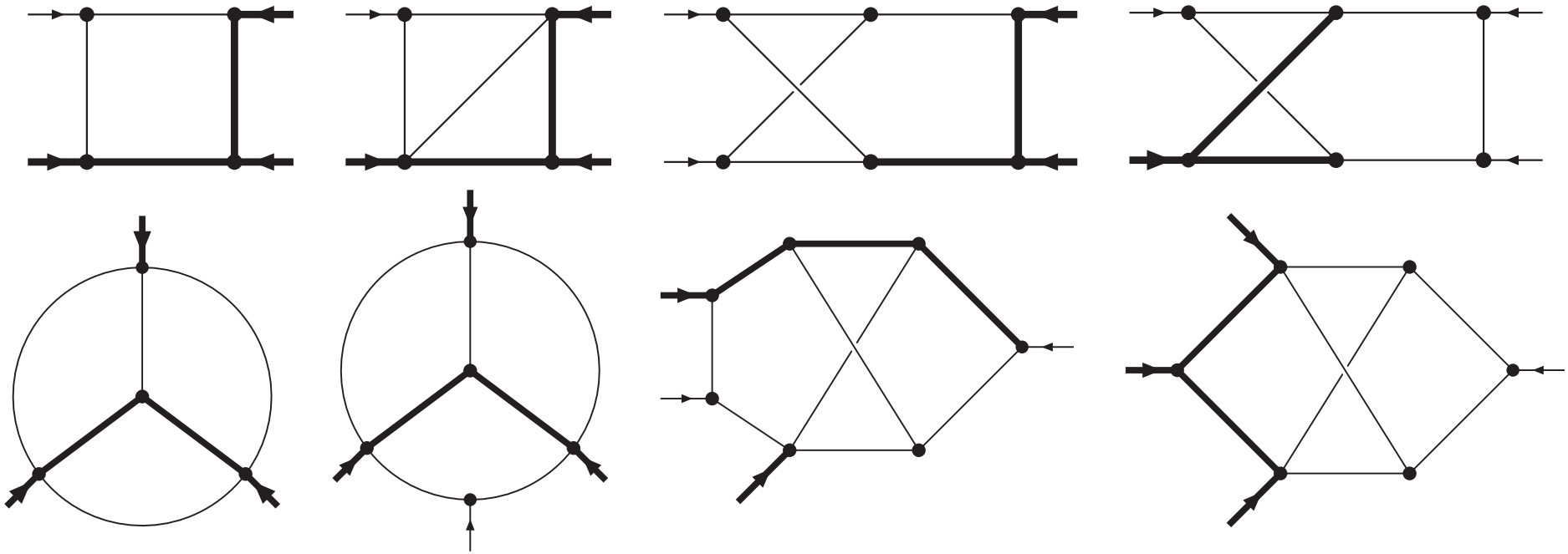
- Counterexamples at three loops, like K_4
- Infinite series: Minors of ladder-boxes and generalizations [16]:



Also with up to two legs off-shell.

Linearly reducible graphs: with massive propagators

Examples:



Notation:

- thin lines: light-like momenta $p_e^2 = 0$, massless propagators $m_e = 0$
- thick lines: arbitrary (different) masses m_e and external momenta p_e^2

Forest polynomials and recursions

Forest polynomials

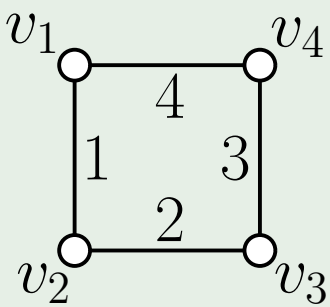
Definition

Spanning forest polynomial $\Phi^{A,B} := \sum_F \prod_{e \notin F} \alpha_e$ over 2-forests F which separate the vertices A and B.

$$f_{12} := \Phi^{\{1,2\},\{3,4\}} \qquad f_3 := \Phi^{\{3\},\{1,2,4\}}$$

$$f_{14} := \Phi^{\{1,4\},\{2,3\}} \qquad f_4 := \Phi^{\{4\},\{1,2,3\}}$$

Example



$$\psi = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \qquad f_{12} = \alpha_2\alpha_4 \qquad f_3 = \alpha_2\alpha_3$$

$$f_{14} = \alpha_1\alpha_3 \qquad f_4 = \alpha_3\alpha_4$$

Forest polynomials

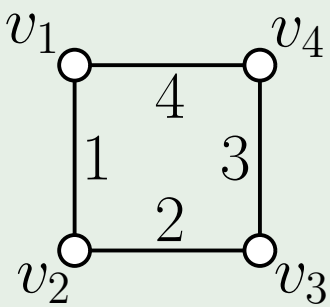
Definition

Spanning forest polynomial $\Phi^{A,B} := \sum_F \prod_{e \notin F} \alpha_e$ over 2-forests F which separate the vertices A and B.

$$f_{12} := \Phi^{\{1,2\},\{3,4\}} \qquad f_3 := \Phi^{\{3\},\{1,2,4\}}$$

$$f_{14} := \Phi^{\{1,4\},\{2,3\}} \qquad f_4 := \Phi^{\{4\},\{1,2,3\}}$$

Example



$$\psi = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \qquad f_{12} = \alpha_2\alpha_4 \qquad f_3 = \alpha_2\alpha_3$$

$$f_{14} = \alpha_1\alpha_3 \qquad f_4 = \alpha_3\alpha_4$$

$$\varphi = \mathcal{F} = (p_1 + p_2)^2 f_{12} + (p_1 + p_4)^2 f_{14} + p_3^2 f_3 + p_4^2 f_4$$

Restricting forest polynomials

Definition

$$F_G(z) := \int_{\mathbb{R}_+^E} \psi_G^{-D/2} \cdot \delta^{(4)}\left(\frac{f}{\psi} - z\right) \prod_{e \in E} \alpha_e^{a_e - 1} \, \mathrm{d}\alpha_e \colon \mathbb{R}_+^4 \longrightarrow \mathbb{R}_+$$

Restricting forest polynomials

Definition

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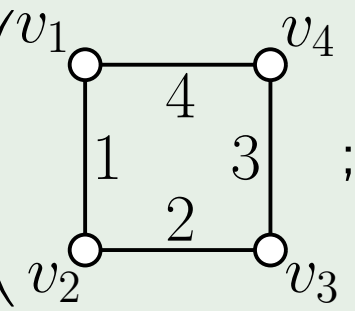
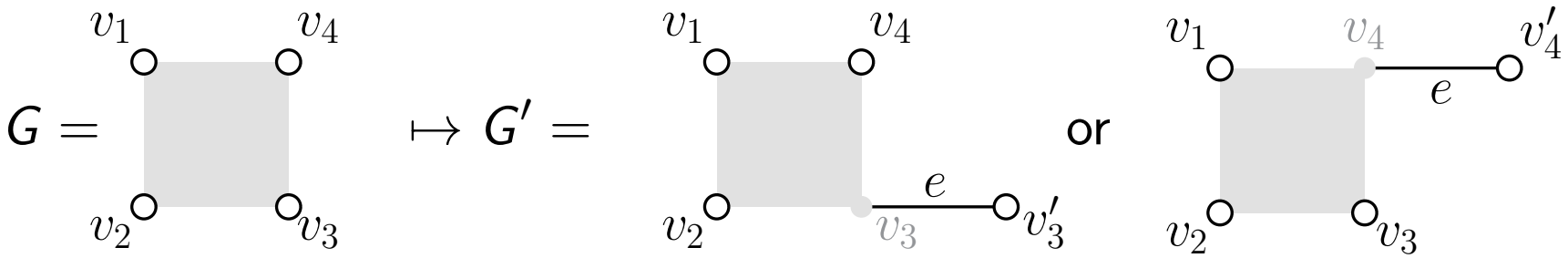
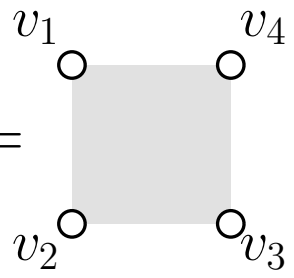
Example ($a_1 = a_2 = a_3 = a_4 = 1$)

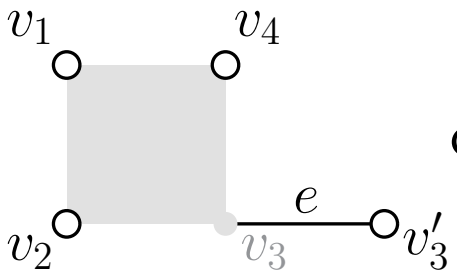
$F\Bigg($  $; z\Bigg) = \begin{cases} \dfrac{1}{z_3 z_4} & (D = 4) \\[2ex] \dfrac{z_{12}}{[\underbrace{z_{12}(z_{14} + z_3 + z_4) + z_3 z_4}_{Q}]^2} & (D = 6) \end{cases}$

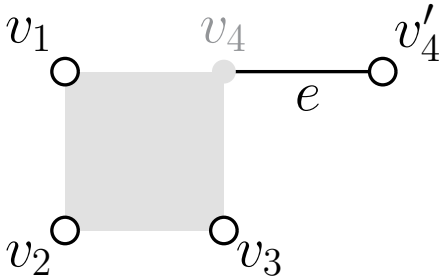
Appending a vertex



Appending a vertex

$G =$ 

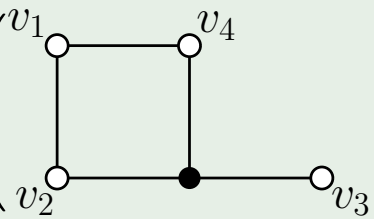
$\mapsto G' =$ 

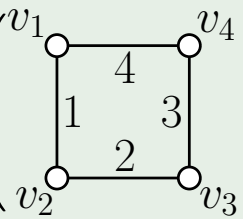
or 

Using $(f_{12}', f_{14}', f_3', f_4', \psi') = (f_{12}, f_{14}, f_3, f_4 + \alpha_e \psi, \psi)$ where $x = \alpha_e$,

$$F_{G'}(z) = \int_0^{z_4} F_G(z_{12}, z_{14}, z_3, z_4 - x) \cdot x^{a_e - 1} \, \mathrm{d}x$$

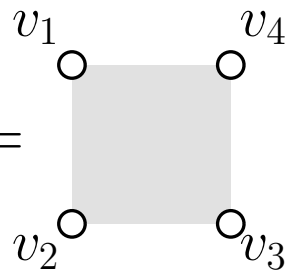
Example ($D = 6$ and $a_e = 1$)

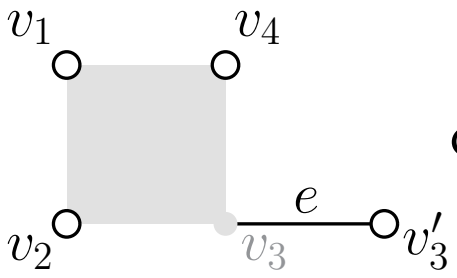
$F\Bigg($ 

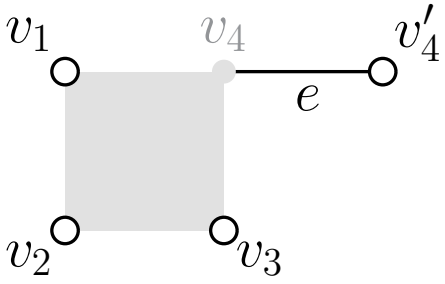
$; z\Bigg) = \int_0^{z_3} F\Bigg($ 

$; z_{12}, z_{14}, z_3', z_4\Bigg) \, \mathrm{d}z_3'$

Appending a vertex

$G =$ 

$\mapsto G' =$ 

or 

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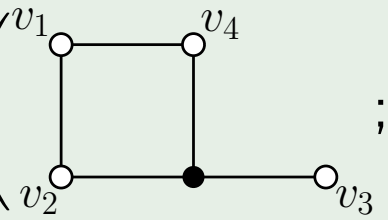
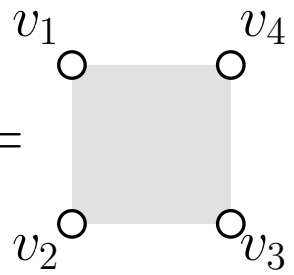
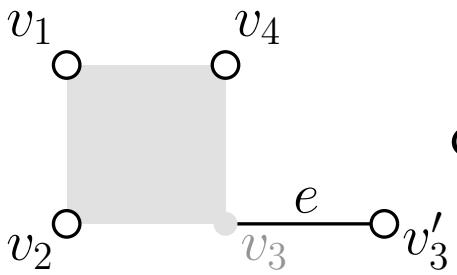
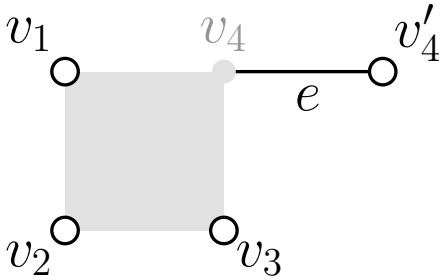
$F\Bigg($ 

$; z\Bigg) = \int_0^{z_3} \frac{z_{12} \, \mathrm{d}z_3'}{[z_{12}(z_{14} + z_3 + z_4) + z_3 z_4]^2} = \frac{z_3}{(z_{14} + z_4) \cdot Q}$

Appending a vertex

$G =$ 

$\mapsto G' =$ 

or 

Using $(f_{12}', f_{14}', f_3', f_4', \psi') = (f_{12}, f_{14}, f_3, f_4 + \alpha_e \psi, \psi)$ where $x = \alpha_e$,

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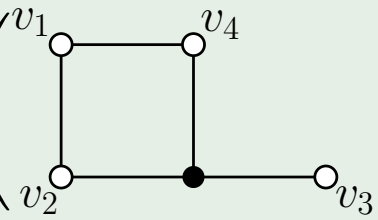
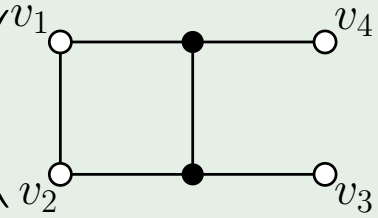
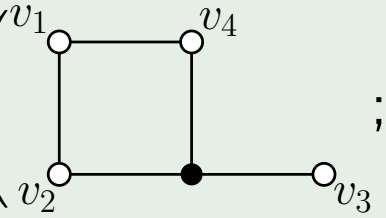
Example ($D = 6$ and $a_e = 1$)

$F\Big($ 

$; z\Big) = \int_0^{z_3} \frac{z_{12} \, \mathrm{d}z_3'}{[z_{12}(z_{14} + z_3 + z_4) + z_3 z_4]^2} = \frac{z_3}{(z_{14} + z_4) \cdot Q}$

$F\Big($ 

$; z\Big) = \int_0^{z_4} F\Big($ 

$; z_{12}, z_{14}, z_3, z_4'\Big) \, \mathrm{d}z_4'$

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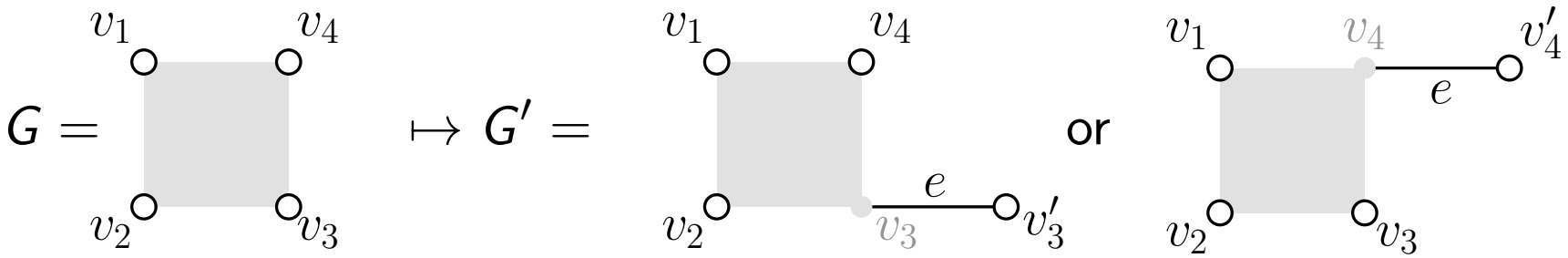


Using $(f_{12}', f_{14}', f_3', f_4', \psi') = (f_{12}, f_{14}, f_3, f_4 + \alpha_e\psi, \psi)$ where $x = \alpha_e$,

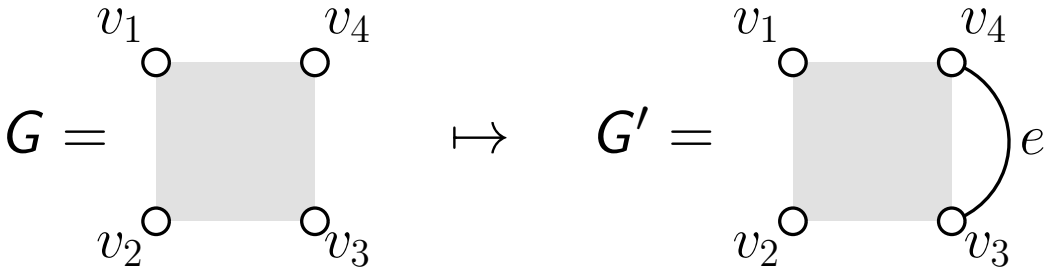
$$F_{G'}(z) = \int_0^{z_4} F_G(z_{12}, z_{14}, z_3, z_4 - x) \cdot x^{a_e - 1} \, \mathrm{d}x$$

Example ($D = 6$ and $a_e = 1$)

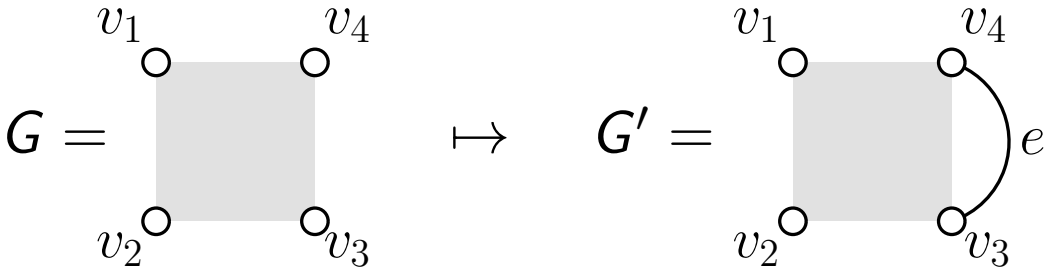
$$F\left(\quad ; z\right) = \int_0^{z_3} \frac{z_{12} \, \mathrm{d}z_3'}{\left[z_{12}\left(z_{14} + z_3 + z_4\right) + z_3 z_4\right]^2} = \frac{z_3}{\left(z_{14} + z_4\right) \cdot Q}$$

$$F\left(\quad ; z\right) = \frac{1}{z_{12} - z_{14}} \ln \frac{z_{12}(z_3 + z_{14})(z_4 + z_{14})}{z_{14} \cdot Q}$$

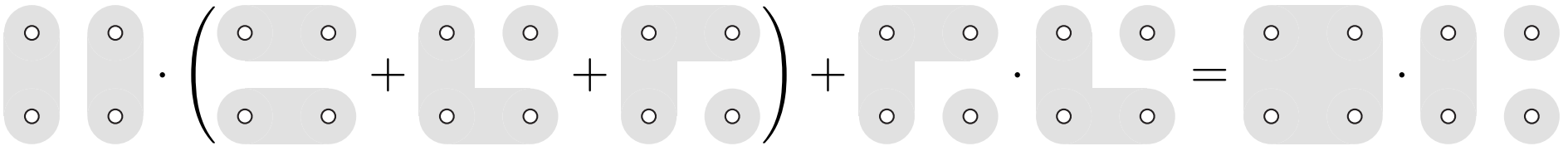
Adding an edge



Adding an edge

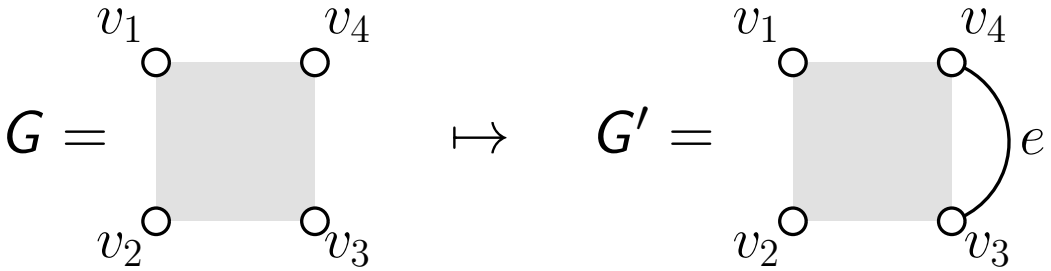


Dodgson-identities between spanning forest polynomials:



$$f_{12}\,(f_{14} + f_3 + f_4) + f_3 f_4 = Q(f) = \psi \cdot \Phi^{\{1,2\},\{3\},\{4\}}$$

Adding an edge



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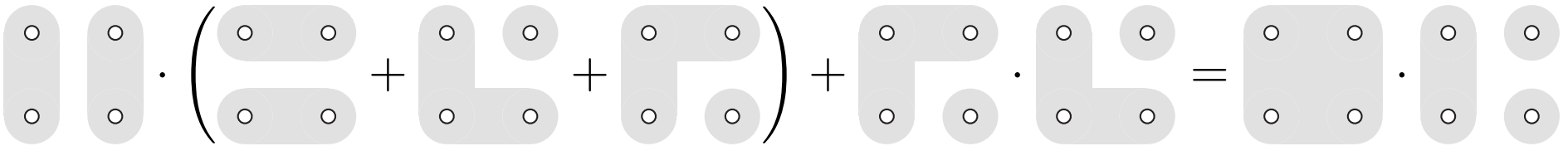
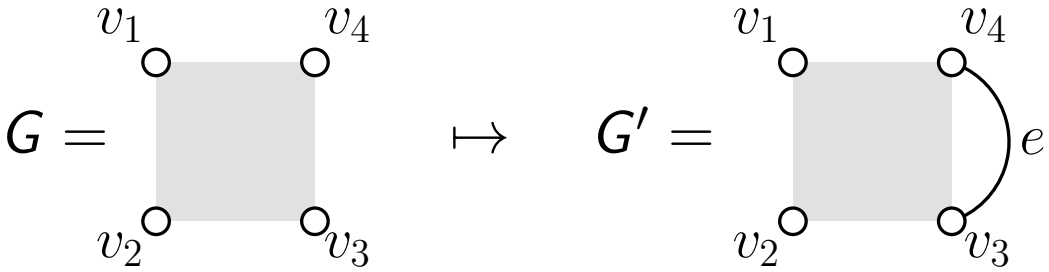


$$f_{12}\,(f_{14} + f_3 + f_4) + f_3 f_4 = Q(f) = \psi \cdot \Phi^{\{1,2\},\{3\},\{4\}}$$

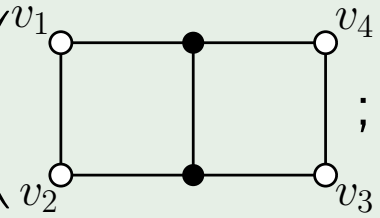
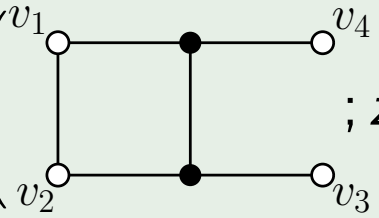
$$F_{G'}(z) = Q^{a_e + \mathrm{sdd} - D} \int_0^{z_{12}} x^{D/2 - a_e - 1} \left[Q^{D/2 - \mathrm{sdd}} \cdot F_G \right]_{z_{12} = z_{12} - x} \mathrm{d}x$$

Adding an edge



$$F_{G'}(z) = Q^{a_e + \mathrm{sdd} - D} \int_0^{z_{12}} x^{D/2 - a_e - 1} \left[Q^{D/2 - \mathrm{sdd}} \cdot F_G \right]_{z_{12} = z_{12} - x} \mathrm{d}x$$

Example ($D = 6$ and $a_e = 1$)

$F\Big($  $; z\Big) = \dfrac{1}{Q^2} \displaystyle\int_0^{z_{12}} F\Big($  $; z_{12} - x, z_{14}, z_3, z_4\Big)\, x \mathrm{d}x$

$$= \frac{z_{12} - z_{14}}{Q^2} \left[\ln \frac{Q}{z_3 z_4} \ln \frac{(z_{14} + z_3)(z_{14} + z_4)}{z_{14}(z_{14} + z_3 + z_4)} - \mathrm{Li}_2 \left(\frac{z_3 z_4 (z_{14} - z_{12})}{z_{14} Q} \right) \right]$$

$$+ \frac{z_{12} - z_{14}}{Q^2} \mathrm{Li}_2 \left(\frac{z_3 z_4}{Q} \right) + \frac{z_{12}}{Q^2} \ln \frac{z_{14} z_3 z_4}{z_{12}(z_{14} + z_3)(z_{14} + z_4)} - \frac{\ln(z_3 z_4 / Q)}{Q(z_{14} + z_3 + z_4)}$$

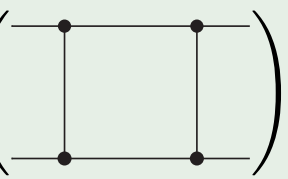
Kinematics

$$\varphi = \mathcal{F} = (p_1 + p_2)^2 f_{12} + (p_1 + p_4)^2 f_{14} + p_3^2 f_3 + p_4^2 f_4$$

Corollary

$$\Phi(G) = \frac{\Gamma(\mathrm{sdd})}{\prod_e \Gamma(a_e)} \int_0^\infty \frac{F_G(z)\,\Omega}{\left[(p_1 + p_2)^2 z_{12} + (p_1 + p_4)^2 z_{14} + p_3^2 z_3 + p_4^2 z_4\right]^{\mathrm{sdd}}}$$

Example ($D = 6$ and $a_e = 1$, $s = (p_1 + p_2)^2$ and $x = (p_1 + p_4)^2/s$)

$s\Phi$ 

$$= \int_0^\infty \frac{\mathrm{d}z_{12}}{z_{12} + x} \int_0^\infty \mathrm{d}z_3 \int_0^\infty \mathrm{d}z_4 \; \frac{z_{12}}{[z_{12}(1 + z_3 + z_4) + z_4 z_3]^2}$$

$$= \int_0^\infty \frac{\mathrm{d}z_{12}}{z_{12} + x} \int_0^\infty \frac{\mathrm{d}z_3}{(1 + z_3)(z_{12} + z_3)}$$

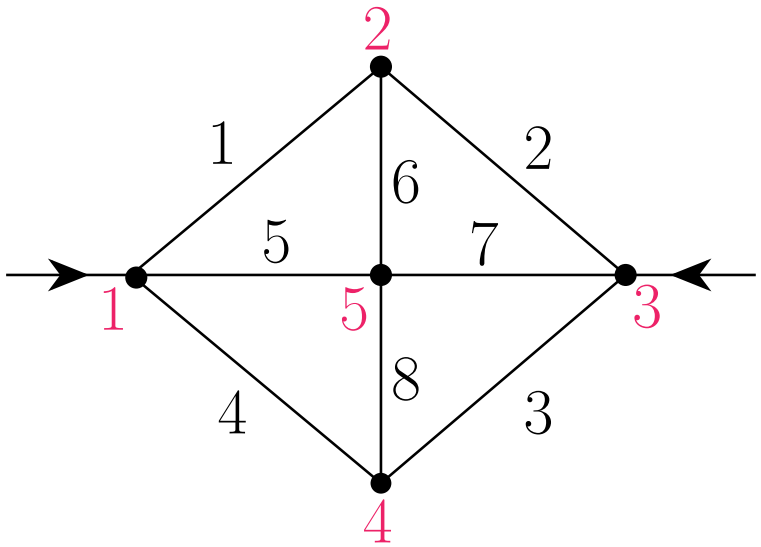
$$= \int_0^\infty \frac{\mathrm{d}z_{12}\ \ln z_{12}}{(z_{12} + x)(z_{12} - 1)} = \frac{\pi^2 + \ln^2 x}{2(1 + x)}$$

Summary

- many linearly reducible graphs with highly non-trivial kinematics
  $\Rightarrow$ arbitrary ε-order, $D|_{\varepsilon=0} \in 2\mathbb{N}$, tensors, $a_e = n_e + \varepsilon\nu_e$
- polynomial reduction determines nature of periods (constants) and letters (alphabet) of final polylogarithms
- changes of variables: so far case-by-case
- parametrizations adapted to graph combinatorics give powerful recursions
- computes individual graphs in contrast to e.g. differential equations method; no boundary terms needed
- also for: phase-space integrals [3, 2], hypergeometric functions, ...
- analytic regularization of divergences via IBP [17]
  $\Rightarrow$ efficient IBP-reduction to quasi-finite masters [22]
- `Maple`^{TM} implementation: `HyperInt` [15]
- current implementation restricted to positive φ/Euclidean region

Thank you.

HyperInt




```
> E := [[1,2],[2,3],[3,4],[4,1],[5,1],[5,2],[5,3],[5,4]]:
> psi := graphPolynomial(E):
> phi := secondPolynomial(E, [[1,1], [3,1]]):
> sdd := nops(E)-(1/2)*4*(4-2*epsilon):
> f := series(psi^(-2+epsilon+sdd)*phi^(-sdd), epsilon=0):
> f := add(coeff(f,epsilon,n)*epsilon^n,n=0..2):
> z := [x[1],x[2],x[6],x[5],x[3],x[4],x[7],x[8]]:
> hyperInt(eval(f,z[-1]=1), z[1..-2]):
> collect(fibrationBasis(%), epsilon);
```

HyperInt

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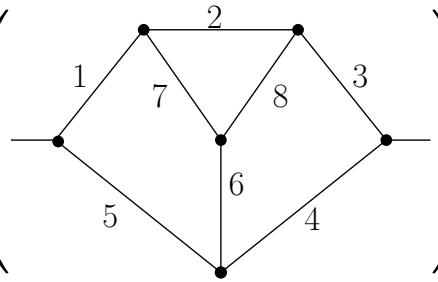
> collect(fibrationBasis(%), epsilon);
```

$$\left(254\zeta_7 + 780\zeta_5 - 200\zeta_2\zeta_5 - 196\zeta_3^2 + 80\zeta_2^3 - \frac{168}{5}\zeta_2^2\zeta_3\right)\varepsilon^2$$
$$+\left(-28\zeta_3^2 + 140\zeta_5 + \frac{80}{7}\zeta_2^3\right)\varepsilon + 20\zeta_5.$$

Divergences in Schwinger parameters

In $D = 4 - 2\varepsilon$ dimensions,

$$\Phi\left(\quad\right) = \Gamma(2+3\varepsilon)\int \frac{\Omega}{\varphi^{2+3\varepsilon}\psi^{-4\varepsilon}}$$



where $\varphi = \alpha_4(\cdots) + \alpha_5(\cdots) + \alpha_4\alpha_5(\cdots)$ vanishes at $\alpha_4 = \alpha_5 = 0$.
Rescale $\alpha_4 \to \lambda\alpha_4$ and $\alpha_5 \to \lambda\alpha_5$, so $\varphi \to \lambda\widetilde{\varphi}$ and $\psi \to \widetilde{\psi}$:

$$\int \frac{\Omega\psi^{4\varepsilon}}{\varphi^{2+3\varepsilon}} = \int \frac{\Omega\psi^{4\varepsilon}}{\varphi^{2+3\varepsilon}} \int_0^\infty \delta(\alpha_4 + \alpha_5 - \lambda)\,\mathrm{d}\lambda$$

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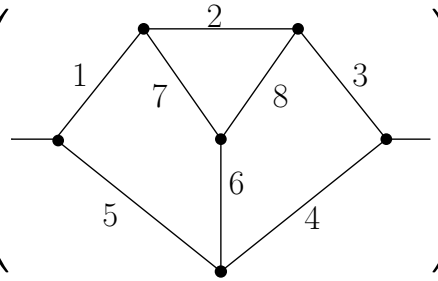

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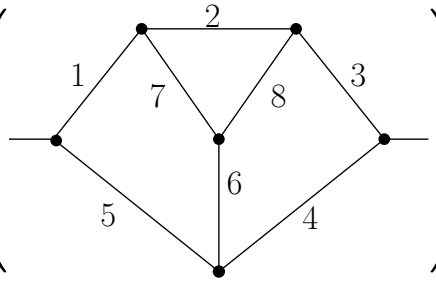
Rescale $\alpha_4 \to \lambda\alpha_4$ and $\alpha_5 \to \lambda\alpha_5$, so $\varphi \to \lambda\widetilde{\varphi}$ and $\psi \to \widetilde{\psi}$:

$$\int \frac{\Omega\psi^{4\varepsilon}}{\varphi^{2+3\varepsilon}} = \int \Omega\ \delta(\alpha_4+\alpha_5-1)\int_0^\infty \frac{\mathrm{d}\lambda}{\lambda^{1+3\varepsilon}}\frac{\widetilde{\psi}^{4\varepsilon}}{\widetilde{\varphi}^{2+3\varepsilon}}$$

Divergences in Schwinger parameters

In $D = 4 - 2\varepsilon$ dimensions,

$$\Phi\left(\quad\right) = \Gamma(2+3\varepsilon)\int \frac{\Omega}{\varphi^{2+3\varepsilon}\psi^{-4\varepsilon}}$$



where $\varphi = \alpha_4(\cdots) + \alpha_5(\cdots) + \alpha_4\alpha_5(\cdots)$ vanishes at $\alpha_4 = \alpha_5 = 0$.
Rescale $\alpha_4 \to \lambda\alpha_4$ and $\alpha_5 \to \lambda\alpha_5$, so $\varphi \to \lambda\widetilde{\varphi}$ and $\psi \to \widetilde{\psi}$:

$$\int \frac{\Omega\psi^{4\varepsilon}}{\varphi^{2+3\varepsilon}} = \int \Omega\ \delta(\alpha_4+\alpha_5-1)\left(\frac{1}{3\varepsilon}\right)\int_0^\infty \frac{\mathrm{d}\lambda}{\lambda^{3\varepsilon}}\left(\frac{\partial}{\partial\lambda}\right)\frac{\widetilde{\psi}^{4\varepsilon}}{\widetilde{\varphi}^{2+3\varepsilon}}$$

Divergences in Schwinger parameters

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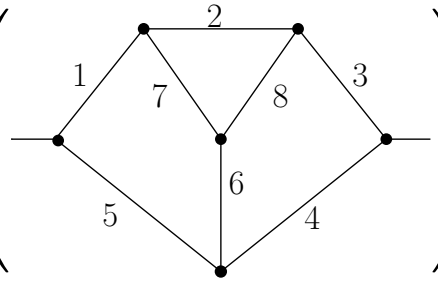

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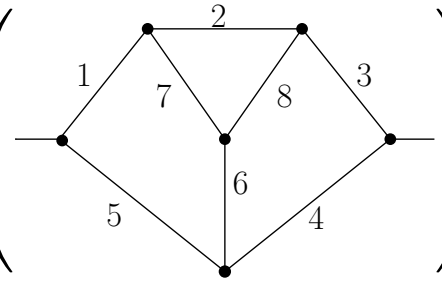

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Numerator monomials correspond to squared propagators:

$$\Phi_D\left(\quad\right) = -\frac{1}{3\varepsilon}\Phi_{D+2}\left(\quad + \quad + \cdots\right)$$

$$+\frac{4}{3}\Phi_{D+2}\left(\quad + \quad + \cdots + 2\quad + 2\quad\right)$$

Regularization of subdivergences

- Subdivergences (IR and UV) manifest as integrand $\sim \lambda^{\omega_\gamma - 1}$ with $\omega_\gamma|_{\varepsilon=0} \leq 0$, when some edges $\alpha_e \sim \lambda$ ($e \in \gamma$) get small jointly
- Well-known power counting:

$$\omega_\gamma = \begin{cases} -\operatorname{sdd}(G/\gamma) & \text{if } \gamma \text{ connects external legs,} \\ \operatorname{sdd}(\gamma) & \text{otherwise.} \end{cases}$$

- Suitable partial integration increments ω_γ
- After finitely many steps, all $\omega_\gamma|_{\epsilon=0} > 0$ (no subdivergences)
  $\Rightarrow$ ε-expansion of integrand gives convergent integrals
- Representation in terms of primitive Feynman integrals, with shifted D and a_e.
- In practice, this creates too many terms and IBP-reduction is necessary

Primitive (quasi-finite) master integrals

Corollary

For any topology, one can choose the master integrals to be quasi-finite (free of subdivergences), given that one allows for shifted dimensions $D + 2$, $D + 4$, ...

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2. General algorithm to find quasi-finite basis [22]
3. Usually it suffices to guess enough quasi-finite integrals

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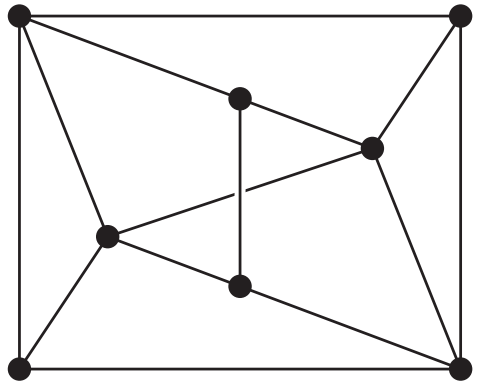
Advantages:

1. ε-expansion of the integrand
2. Directly suitable for numeric quadrature
3. No splitting into non-Feynman integrals like sector decomposition

Massless ϕ^4 theory: primitive sixth roots of unity

$P_{7,11} =$ 

$P_{7,11}$ is not linearly reducible: After integrating ten variables, denominator

$$\begin{aligned} d_{10} = {} & \alpha_2\alpha_4^2\alpha_1 + \alpha_2\alpha_4^2\alpha_3 - \alpha_1\alpha_2\alpha_3\alpha_4 + \alpha_2^2\alpha_4\alpha_1 + \alpha_2^2\alpha_4\alpha_3 \\ & - 2\alpha_2\alpha_3^2\alpha_4 - \alpha_2^2\alpha_3^2 - 2\alpha_2^2\alpha_3\alpha_1 - 2\alpha_2\alpha_3^2\alpha_1 - \alpha_3^2\alpha_4^2 \\ & - 2\alpha_3^2\alpha_4\alpha_1 - \alpha_2^2\alpha_1^1 - 2\alpha_2\alpha_3\alpha_1^2 - \alpha_3^2\alpha_1^2. \end{aligned}$$

Changing variables $\alpha_3 = \frac{\alpha_3'\alpha_1}{\alpha_1+\alpha_2+\alpha_4}$, $\alpha_4 = \alpha_4'(\alpha_2 + \alpha_3')$ and $\alpha_1 = \alpha_1'\alpha_4'$,

$$d_{10}' = (\alpha_2 + \alpha_3')(\alpha_2 + \alpha_2\alpha_4' - \alpha_1')(\alpha_1'\alpha_4' + \alpha_2 + \alpha_2\alpha_4' + \alpha_3'\alpha_4')$$

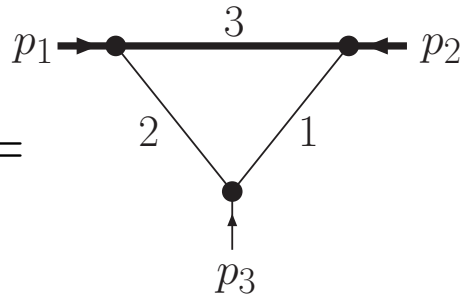
factors linearly and $\alpha_1', \alpha_3', \alpha_4'$ can be integrated ($\alpha_2 = 1$).
The final integrand is $\mathrm{HPL}(\alpha_1)/(1 - \alpha_1 + \alpha_1^2)$ and gives *not a multiple zeta value*, but a polylogarithm at sixth roots of unity.

$\sqrt{3}\,\mathcal{P}(P_{7,11})$

$$
\begin{aligned}
&= \operatorname{Im}\left(\tfrac{19\,285}{6}\zeta_9\operatorname{Li}_2 - \tfrac{1029}{2}\zeta_7\operatorname{Li}_4 + 240\zeta_3^2(9\operatorname{Li}_{2,3} - 7\zeta_3\operatorname{Li}_2)\right) - \tfrac{93\,824}{9675}\pi^3\zeta_{3,5} \\
&+ \tfrac{2592}{215}\operatorname{Im}\Big(36\operatorname{Li}_{2,2,2,5} + 27\operatorname{Li}_{2,2,3,4} + 9\operatorname{Li}_{2,2,4,3} + 9\operatorname{Li}_{2,3,2,4} + 3\operatorname{Li}_{2,3,3,3} \\
&\qquad - 43\zeta_3(\operatorname{Li}_{2,3,3} + 3\operatorname{Li}_{2,2,4})\Big) - \tfrac{96\,393\,596\,519\,864\,341\,538\,701\,979}{790\,371\,465\,315\,684\,594\,157\,620\,000}\pi^{11} \\
&+ \tfrac{216}{14\,755\,731\,798\,995}\operatorname{Im}\Big(2\,539\,186\,130\,125\,890\operatorname{Li}_8\zeta_3 - 1\,269\,593\,065\,062\,945\operatorname{Li}_{2,9} \\
&\qquad - 413\,965\,317\,054\,502\operatorname{Li}_6\zeta_5 - 996\,412\,983\,391\,539\operatorname{Li}_{3,8} \\
&\qquad - 546\,306\,741\,059\,841\operatorname{Li}_{4,7} - 156\,228\,639\,992\,955\operatorname{Li}_{5,6}\Big) \\
&+ \tfrac{2592}{10\,945\,435}\pi^2\operatorname{Im}\left(287\,205\operatorname{Li}_{2,7} - 574\,410\operatorname{Li}_6\zeta_3 + 55\,687\operatorname{Li}_{4,5} + 168\,941\operatorname{Li}_{3,6}\right) \\
&+ \pi\left(\tfrac{11\,613\,751}{9030}\zeta_5^2 + \tfrac{267\,067}{602}\zeta_{3,7} - \tfrac{31\,104}{215}\operatorname{Re}(3\operatorname{Li}_{4,6} + 10\operatorname{Li}_{3,7})\right)
\end{aligned}
$$

Abbreviation: $\operatorname{Li}_{n_1,\ldots,n_r} := \operatorname{Li}_{n_1,\ldots,n_r}(e^{i\pi/3})$

Divergences in Schwinger parameters

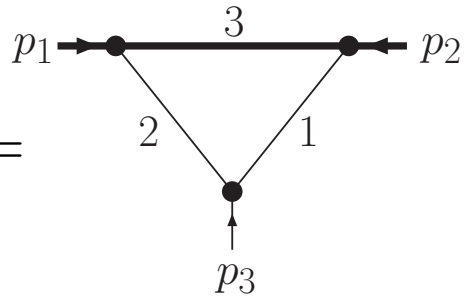
$G =$ 

$$\psi = \alpha_1 + \alpha_2 + \alpha_3$$
$$\varphi = \alpha_3 \left(m^2 \psi + p_1^2 \alpha_2 + p_2^2 \alpha_1 \right)$$

Linearly reducible, but in $D = 4 - 2\varepsilon$, sdd $= 1 + \varepsilon$ such that $\int_0^\infty \mathrm{d}\alpha_3$ diverges at the lower boundary when $\varepsilon \to 0$:

$$\int \frac{\Omega}{(\alpha_1 + \alpha_2 + \alpha_3)^{1-2\varepsilon} \left[m^2(\alpha_1 + \alpha_2 + \alpha_3) + p_1^2 \alpha_2 + p_2^2 \alpha_1 \right]^{1+\varepsilon} \alpha_3^{1+\varepsilon}}$$

Divergences in Schwinger parameters

$G =$ 

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$$\int \frac{\Omega}{(\alpha_1+\alpha_2+\alpha_3)^{1-2\varepsilon}\left[m^2(\alpha_1+\alpha_2+\alpha_3)+p_1^2\alpha_2+p_2^2\alpha_1\right]^{1+\varepsilon}\alpha_3^{1+\varepsilon}}$$

Example (Analytic regularization of a single logarithmic divergence)

Let $\tilde{\phi} := \varphi/\alpha_3 = m^2\psi + p_2^2\alpha_1 + p_1^2\alpha_2$ and integrate by parts:

$$\int \frac{\Omega}{\psi^{1-2\varepsilon}\tilde{\varphi}^{1+\varepsilon}\alpha_3^{\varepsilon}} = \left.\frac{\alpha_3^{-\varepsilon}}{-\varepsilon\psi^{1-2\varepsilon}\tilde{\varphi}^{1+\varepsilon}}\right|_{\alpha_3=0}^{\infty} + \frac{1}{\varepsilon}\cdot\int \frac{\Omega}{\alpha_3^{\varepsilon}}\frac{\partial}{\partial\alpha_3}\psi^{-1+2\varepsilon}\tilde{\varphi}^{-1-\varepsilon}$$
$$= \frac{1}{\varepsilon}\cdot\int \frac{\Omega\alpha_3}{\psi^{1-2\varepsilon}\varphi^{1+\varepsilon}}\left[\frac{2\varepsilon-1}{\psi} - \frac{(1+\varepsilon)\alpha_3 m^2}{\varphi}\right] \text{ when } \varepsilon < 0.$$

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