Multiple polylogarithms and Feynman integrals

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Outline:

1. Multiple polylogarithms in several variables

2. Linear reducibility of Symanzik polynomials

3. The two-loop sunrise graph with arbitrary masses

   - a case beyond multiple polylogarithms

1. Multiple polylogarithms in several variables

Motivation:

Iterated integrals are crucial for the computation of Feynman integrals

One frequently computes with

- classical polylogarithms, e.g.

$$\mathrm{Li}_2(z) = -\int_0^z \frac{dx'}{x'} \ln(1-x') = \int_0^z \frac{dx'}{x'} \int_0^{x'} \frac{dx}{1-x}$$

- harmonic polylogarithms, e.g.

$$\begin{aligned} H(-1,0,0,1;z) &= \int_0^z \frac{dx'''}{1+x'''} \mathrm{Li}_3(x''') \\ &= \int_0^z \frac{dx'''}{1+x'''} \int_0^{x'''} \frac{dx''}{x''} \int_0^{x''} \frac{dx'}{x'} \int_0^{x'} \frac{dx}{1-x} \end{aligned}$$

General definition for iterated integrals

Let

- k be a field (either $\mathbb{R}$ or $\mathbb{C}$),
- M a smooth manifold over k,
- $\gamma : [0, 1] \to M$ a smooth path on M,
- $\omega_1, \dots, \omega_n$ smooth differential 1-forms on M,
- $\gamma^\star(\omega_i) = f_i(t)dt$ the pull-back of ω_i to $[0, 1]$

Def.: The *iterated integral* of $\omega_1, \dots, \omega_n$ along γ is

$$\int_\gamma \omega_n \dots \omega_1 = \int_{0 \le t_1 \le \dots \le t_n \le 1} f_n(t_n)dt_n \dots f_1(t_1)dt_1 .$$

We use the term *iterated integral* for k-linear combinations of such integrals.

We obtain a different classes of functions by choosing different finite sets of 1-forms Ω.

- $\Omega_1 = \left\{ \frac{dt}{t}, \frac{dt}{t-1} \right\}$, $\omega_0 \equiv \frac{dt}{t}$, $\omega_1 \equiv \frac{dt}{t-1}$
  - **classical polylogarithms**:

$$\mathrm{Li}_n(z) = \int_\gamma \underbrace{\omega_0 \ldots \omega_0}_{n-1\,\mathrm{times}} \omega_1 = \int_{0 \le t_1 \le \ldots \le t_n \le 1} \frac{dt_n}{t_n} \ldots \frac{dt_2}{t_2} \frac{z dt_1}{1 - z t_1}$$

  - **multiple polylogarithms in one variable**:

$$\mathrm{Li}_{n_1, \ldots, n_r}(z) = (-1)^r \int_\gamma \underbrace{\omega_0 \ldots \omega_0}_{n_r - 1} \omega_1 \ldots \underbrace{\omega_0 \ldots \omega_0}_{n_1 - 1} \omega_1$$

where γ a smooth path in $\mathbb{C} \backslash \{0, 1\}$ with end-point z

- $\Omega_n^{\mathrm{Hyp}} = \left\{ \frac{dt_1}{t_1}, \frac{dt_1}{t_1-1}, \frac{t_2\,dt_1}{t_1 t_2 - 1}, \ldots, \frac{\left(\prod_{i=2}^n t_i\right) dt_1}{\prod_{i=1}^n t_i - 1} \right\}$: **hyperlogarithms**
  (Poincare, Kummer 1840, Lappo-Danilevsky 1953)
  special cases:
  - for $n = 2$, $t_2 = -1$ this is $\left\{ \frac{dt_1}{t_1}, \frac{dt_1}{t_1-1}, \frac{dt_1}{t_1+1} \right\}$: **harmonic polylogarithms**
    (Remiddi, Vermaseren 1999)
  - for $n = 3$, $t_2 = -\frac{1}{z}$, $t_3 = \frac{z}{z-1}$ this is $\left\{ \frac{dt_1}{t_1}, \frac{dt_1}{t_1-1}, \frac{dt_1}{t_1+z}, \frac{dt_1}{t_1+z-1} \right\}$:
    two-dimensional harmonic polylogarithms (Gehrmann, Remiddi '01)

Other classes of iterated integrals used in physics: **cyclotomic harmonic polylogarithms** (Ablinger, Blümlein, Schneider '11), **multiple polylogarithms** (Goncharov '01)

We want to construct a class closely related to Goncharov's multiple polylogarithms with particularly good properties.

For any positive integer n we consider the set

$$\Omega_n = \left\{ \frac{dt_1}{t_1}, \dots, \frac{dt_n}{t_n}, \frac{d\left(\prod_{a \le i \le b} t_i\right)}{\prod_{a \le i \le b} t_i - 1} \text{ where } 1 \le a \le b \le n \right\}$$

Examples:

$$\Omega_1 = \left\{ \frac{dt_1}{t_1}, \frac{dt_1}{t_1 - 1} \right\} \text{ (see multiple polylogs in one variable)}$$

$$\Omega_2 = \left\{ \frac{dt_1}{t_1}, \frac{dt_2}{t_2}, \frac{dt_1}{t_1 - 1}, \frac{dt_2}{t_2 - 1}, \frac{t_1\, dt_2 + t_2\, dt_1}{t_1 t_2 - 1} \right\}$$

Note:

- They involve all $dt_1, dt_2, \dots, dt_n$.
- They are of the type $\frac{df}{f}$ with $f = \prod_i t_i - 1$.

From this Ω_n we want to construct iterated integrals which are *homotopy invariant*.

Def.: Smooth paths γ_1, γ_2 on M are *homotopic* if their end-points coincide and if γ_1 can be continuously transformed into γ_2.

Def.: An iterated integral is called *homotopy invariant* if it satisfies

$$\int_{\gamma_1} \omega_n ... \omega_1 = \int_{\gamma_2} \omega_n ... \omega_1$$

for homotopic γ_1, γ_2.

By such integrals we obtain function of **variables given only by the end-points** of paths.

When is an iterated integral homotopy invariant?

Consider tensor products $\omega_1 \otimes \ldots \otimes \omega_m \equiv [\omega_1|\ldots|\omega_m]$ over $\mathbb{Q}$.

Define an operator D by

$$D\left([\omega_1|\ldots|\omega_m]\right) = \sum_{i=1}^{m}[\omega_1|\ldots|\omega_{i-1}|d\omega_i|\omega_{i+1}|\ldots\omega_m] + \sum_{i=1}^{m-1}[\omega_1|\ldots|\omega_{i-1}|\omega_i \wedge \omega_{i+1}|\ldots|\omega_m].$$

Def.: A $\mathbb{Q}-$linear combination of tensor products

$$\xi = \sum_{l=0}^{m} \sum_{i_1, \ldots, i_l} c_{i_1, \ldots, i_l}[\omega_{i_1}|\ldots|\omega_{i_l}], \; c_{i_1, \ldots, i_l} \in \mathbb{Q}$$

is called *integrable word* if

$$D(\xi) = 0.$$

Consider the integration map

$$\sum_{l=0}^{m} \sum_{i_1, \dots, i_l} c_{i_1, \dots, i_l} [\omega_{i_1} | \dots | \omega_{i_l}] \mapsto \sum_{l=0}^{m} \sum_{i_1, \dots, i_l} c_{i_1, \dots, i_l} \int_\gamma \omega_{i_1} \dots \omega_{i_l}$$

Theorem (Chen '77): Under certain conditions on Ω this map is an isomorphism from *integrable words* to *homotopy invariant iterated integrals*.

Construction of our class of homotopy invariant functions:

- Construct the integrable words of 1-forms in Ω_n. (Using the 'symbol map'.)
- By the integration map obtain the set of **multiple polylogarithms in several variables** $\mathcal{B}(\Omega_n)$.

Properties of $\mathcal{B}(\Omega_n)$ (Brown '05):

- They are well-defined functions of n variables, corresponding to end-points of paths.
- On these functions, functional relations turn into algebraic identities.
- Via the 'symbol map' we have a decomposition and an explicit basis.
- $\mathcal{B}(\Omega_n)$ is closed under taking primitives.
- Let $\mathcal{Z}$ be the $\mathbb{Q}$-vector space of multiple zeta values. The limits at 0 and 1 of functions in $\mathcal{B}(\Omega_n)$ are $\mathcal{Z}$-linear combinations of elements in $\mathcal{B}(\Omega_{n-1})$.

Consequence:

Let F_n be the vector space of rational functions with denominators in $\left\{t_1, \ldots, t_n, \prod_{a\leq i\leq b} t_i - 1\right\}, 1 \leq a \leq b \leq n.$

Consider integrals of the type

$$\int_0^1 dt_n \sum_j f_j \beta_j \text{ with } f_j \in F_n,\ \beta_j \in \mathcal{B}(\Omega_n).$$

We can compute such integrals. The results are $\mathcal{Z}$-linear combinations of elements in $\mathcal{B}(\Omega_{n-1})$, multiplied by elements in F_{n-1}.

Concept: Map *Feynman integrals* to integrals of this type and evaluate them.

When is this possible?

2. Linear reducibility of Symanzik polynomials

Feynman integrals:

$$I_G(\epsilon, \Lambda_G) = \frac{\Gamma(\nu - LD/2)}{\prod_{j=1}^N \Gamma(\nu_j)} \int_{x_j \geq 0} \left(\prod_{i=1}^N dx_i x_i^{\nu_i - 1} \right) \delta\left(1 - \sum_{i=1}^N x_i\right) \frac{\mathcal{U}_G^{\nu-(L+1)D/2}}{(\mathcal{F}_G(\Lambda_G))^{\nu - LD/2}},$$

where $\nu = \sum_{j=1}^N \nu_j$.

Symanzik polynomials:

$$\begin{aligned}
\mathcal{U}(G) &= \sum_{\text{spanning trees } T \text{ of } G} \prod_{\text{edges } \notin T} x_i \\
\mathcal{F}_0(G) &= -\sum_{\text{spanning 2-forests } (T_1,\, T_2)} \left(\prod_{\text{edges } \notin (T_1,\, T_2)} x_i \right) \left(\sum_{\text{edges } \notin (T_1,\, T_2)} q_i \right)^2, \\
\mathcal{F}(G) &= \mathcal{F}_0(G) + \mathcal{U}(G) \sum_{i=1}^N x_i m_i^2.
\end{aligned}$$

Example 1: Vacuum graphs with $\nu = 2L$ and $D = 4$:

$$\int_{x_j \geq 0} \left(\prod_{i=1}^{N} dx_i x_i^{\nu_i - 1} \right) \delta \left(1 - \sum_{i=1}^{N} x_i \right) \frac{1}{\mathcal{U}_G^2}$$

Example 2: Sunrise graph with $\nu = L + 1$ and $D = 2$:

$$\int_{x_j \geq 0} \left(\prod_{i=1}^{N} dx_i x_i^{\nu_i - 1} \right) \delta \left(1 - \sum_{i=1}^{N} x_i \right) \frac{1}{\mathcal{F}_G \left(\Lambda_G \right)}$$

Assume a **finite** integral whose integrand is given by one or both of the **Symanzik polynomials**.

Try to integrate out the Feynman parameters iteratively.

- After integration over x_i consider the set S_i of polynomials in the denominator and in arguments of multiple polylogs.
  Check that all polynomials in S_i are **linear in a next Feynman parameter** x_{i+1}.
- Map the integral over x_{i+1} to an integral over t_n of the form

$$\int_0^1 dt_n \sum_j f_j \beta_j \text{ with } f_j \in F_n,\ \beta_j \in \mathcal{B}(\Omega_n)$$

and integrate over t_n.

The linear reduction algorithm gives an upper bound for the S_i.

Linear reduction algorithm (Brown '08)
Input:

- a set of polynomials S,
- a sequence of Feynman parameters $x_{r_1}, x_{r_2}, \ldots, x_{r_n}$

Output: a sequence of sets of polynomials $S_1, S_2, \ldots, S_n$

S_k contains the polynomials which we expect in the integrand after integrating out the first k parameters.
If all polynomials in $S_{[r_1, \ldots, r_k]}$ are linear in $x_{r_{k+1}}$, the next integration can be done.

Def.: A set S is *linearly reducible* if there is an ordering $(x_{r_1}, x_{r_2}, \ldots, x_{r_n})$ such that for all $1 \leq k \leq n$ every polynomial in $S_{[r_1, \ldots, r_k]}$ is linear in $x_{r_{k+1}}$.

If this is true for $\{\mathcal{U}_G, \mathcal{F}_G\}$ we say that the Feynman graph G is linearly reducible.

Consider the deletion and contraction of edges:

$G \backslash e$: graph obtained from *deleting* edge e in G
$G//e$: graph obtained from *contracting* edge e in G

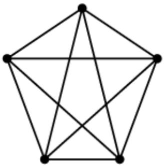
Deletion and contraction of different edges commute. $\Rightarrow$ consider $G \backslash D//C$ where C, D are disjoint sets of edges
Any such graph is called *minor* of G.

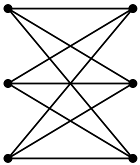
Def.: A set $\mathcal{G}$ of graphs is called *minor-closed* if for each $G \in \mathcal{G}$ all minors belong to $\mathcal{G}$ as well.

Let $\mathcal{H}$ be a finite set of graphs.
Define $\mathcal{G}_{\mathcal{H}}$ to be the set of graphs **whose minors do not belong** to $\mathcal{H}$.
Then the graphs in $\mathcal{H}$ are called *forbidden minors* of $\mathcal{G}_{\mathcal{H}}$. The set $\mathcal{G}_{\mathcal{H}}$ is minor-closed.

Theorem (Robertson and Seymour): Any minor-closed set of graphs can be defined by a finite set of forbidden minors.

Example:
Let $\mathcal{G}$ be the set of all **planar graphs**. This set is minor closed.
It can be defined as the set of all graphs which have neither K_5 nor $K_{3,3}$ as a minor.





Theorem (Brown '09, CB and Lüders '13): The set of linearly reducible Feynman graphs is minor-closed.

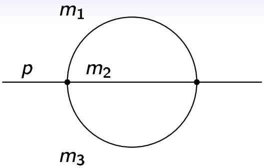
We should search for the forbidden minors.

A first case study (with M. Lüders):

- Let Λ be the set of massless Feynman graphs with four on-shell legs. (On-shell condition: $p_i^2 = 0,\ i = 1, \ldots, 4$)
- At two loops we find all graphs to be linearly reducible.
- At three loops we find first forbidden minors.
- Four loops are running on our computers and confirm the forbidden three-loop minors so far.

Fazit: A classification w.r.t. linearly reducibility is possible by forbidden minors. It gives a hint on the more difficult problem of which graphs evaluate to multiple polylogarithms and which don't.

3. The two-loop sunrise graph with arbitrary masses



In $D = 2$ dimensions we obtain the finite Feynman integral

$$S_{D=2}(t) = \int_\sigma \frac{\omega}{\mathcal{F}_G},$$

with

$$\omega = x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2$$

$$\mathcal{F}_G\left(t,\, m_1^2,\, m_2^2,\, m_3^2\right) = -x_1 x_2 x_3 t + (x_1 x_2 + x_2 x_3 + x_1 x_3)(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2), \quad t = p^2,$$

$$\sigma = \left\{ [x_1 : x_2 : x_3] \in \mathbb{P}^2 | x_i \geq 0,\ i = 1, 2, 3 \right\}$$

As $\mathcal{F}_G$ is not linear in any x_i, the graph is not linearly reducible.

(Incomplete) history of sunrises:
Equal mass case:

- Broadhurst, Fleischer, Tarasov (1993): result with hypergeometric functions
- Groote, Pivovarov (2000): Cutkosky rules $\Rightarrow$ imaginary part expressed by elliptic integrals
- Laporta, Remiddi (2004): solving a second-order differential equation $\Rightarrow$ result by integrals over elliptic integrals

Arbitrary mass case:

- Berends, Buza, Böhm, Scharf (1994): result with Lauricella functions
- Caffo, Czyz, Laporta, Remiddi (1998): system of four first-order differential equations (and numerical solutions)
- Groote, Körner, Pivovarov (2005): integral representations involving Bessel functions
- Müller-Stach, Weinzierl, Zayadeh (2012): one second-order differential equation

Our goal: Solve the new differential equation (as Laporta and Remiddi did for equal masses) and obtain a result involving elliptic integrals

A result for D dimensions is known from Berends, Buza, Böhm and Scharf (1994):

$$S_D(t) =$$

$$(-t)^{D-3} \left(\frac{\Gamma(3-D)\Gamma(\frac{D}{2}-1)^3}{\Gamma(\frac{3}{2}D-3)} F_C \left(3-D,\, 4-\tfrac{3}{2}D;\, 2-\tfrac{D}{2},\, 2-\tfrac{D}{2},\, 2-\tfrac{D}{2};\, \frac{m_1^2}{t},\, \frac{m_2^2}{t},\, \frac{m_3^2}{t}\right) \right.$$

$$\frac{\Gamma(2-\frac{D}{2})\Gamma(1-\frac{D}{2})\Gamma(\frac{D}{2}-1)^2}{\Gamma(D-2)} \left(F_C \left(3-D,\, 2-\tfrac{D}{2};\, \tfrac{D}{2},\, 2-\tfrac{D}{2},\, 2-\tfrac{D}{2};\, \frac{m_1^2}{t},\, \frac{m_2^2}{t},\, \frac{m_3^2}{t}\right) \left(-\frac{m_1^2}{t}\right)^{\frac{D}{2}-1} \right.$$

$$+F_C \left(3-D,\, 2-\tfrac{D}{2};\, 2-\tfrac{D}{2},\, \tfrac{D}{2},\, 2-\tfrac{D}{2};\, \frac{m_1^2}{t},\, \frac{m_2^2}{t},\, \frac{m_3^2}{t}\right) \left(-\frac{m_2^2}{t}\right)^{\frac{D}{2}-1}$$

$$\left. + F_C \left(3-D,\, 2-\tfrac{D}{2};\, 2-\tfrac{D}{2},\, 2-\tfrac{D}{2},\, \tfrac{D}{2};\, \frac{m_1^2}{t},\, \frac{m_2^2}{t},\, \frac{m_3^2}{t}\right) \left(-\frac{m_3^2}{t}\right)^{\frac{D}{2}-1} \right)$$

$$+\Gamma(1-\tfrac{D}{2})^2 \left(F_C \left(1,\, 2-\tfrac{D}{2};\, \tfrac{D}{2},\, \tfrac{D}{2},\, 2-\tfrac{D}{2};\, \frac{m_1^2}{t},\, \frac{m_2^2}{t},\, \frac{m_3^2}{t}\right) \left(\frac{m_1^2 m_2^2}{t^2}\right)^{\frac{D}{2}-1} \right.$$

$$+F_C \left(1,\, 2-\tfrac{D}{2};\, \tfrac{D}{2},\, 2-\tfrac{D}{2},\, \tfrac{D}{2};\, \frac{m_1^2}{t},\, \frac{m_2^2}{t},\, \frac{m_3^2}{t}\right) \left(\frac{m_1^2 m_3^2}{t^2}\right)^{\frac{D}{2}-1}$$

$$\left.\left. + F_C \left(1,\, 2-\tfrac{D}{2};\, 2-\tfrac{D}{2},\, \tfrac{D}{2},\, \tfrac{D}{2};\, \frac{m_1^2}{t},\, \frac{m_2^2}{t},\, \frac{m_3^2}{t}\right) \left(\frac{m_2^2 m_3^2}{t^2}\right)^{\frac{D}{2}-1} \right)\right)$$

with the Lauricella function

$$F_C(a_1, a_2; b_1, b_2, b_3; x_1, x_2, x_3) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} \frac{(a_1)_{j_1+j_2+j_3} (a_2)_{j_1+j_2+j_3}}{(b_1)_{j_1} (b_2)_{j_2} (b_3)_{j_3}} \frac{x_1^{j_1} x_2^{j_2} x_3^{j_3}}{j_1! j_2! j_3!}$$

and the Pochhammer symbol $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$

Using Euler-Zagier sums $Z_1(n) = \sum_{j=1}^{n} \frac{1}{j}$, $Z_{11}(n) = \sum_{j=1}^{n} \frac{1}{j} Z_1(j-1)$ we can expand the result in $D = 2$ and obtain:

$$S_{D=2}(t) = -\frac{1}{t} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} \left(\frac{j_{123}!}{j_1! j_2! j_3!} \right)^2 \left(\frac{m_1^2}{t} \right)^{j_1} \left(\frac{m_2^2}{t} \right)^{j_2} \left(\frac{m_3^2}{t} \right)^{j_3}$$

$$(12 Z_{11}(j_{123}) + 6 Z_1(j_{123}) Z_1(j_{123}) - 8 Z_1(j_{123}) \left(Z_1(j_1) + Z_1(j_2) + Z_1(j_3) \right)$$

$$4 \left(Z_1(j_1) Z_1(j_2) + Z_1(j_2) Z_1(j_3) + Z_1(j_3) Z_1(j_1) \right) +$$
$$2 \left(2 Z_1(j_{123}) - Z_1(j_2) - Z_1(j_3) \right) \ln \left(-\frac{m_1^2}{t} \right) + 2 \left(2 Z_1(j_{123}) - Z_1(j_3) - Z_1(j_1) \right) \ln \left(-\frac{m_2^2}{t} \right)$$

$$+2 \left(2 Z_1(j_{123}) - Z_1(j_1) - Z_1(j_2) \right) \ln \left(-\frac{m_3^2}{t} \right)$$

$$+ \ln \left(-\frac{m_1^2}{t} \right) \ln \left(-\frac{m_2^2}{t} \right) + \ln \left(-\frac{m_2^2}{t} \right) \ln \left(-\frac{m_3^2}{t} \right) + \ln \left(-\frac{m_1^2}{t} \right) \ln \left(-\frac{m_3^2}{t} \right))$$

We obtain a five-fold nested sum.
Can we express the integral by iterated integrals instead?

Start from the second order differential equation:

$$\left(p_0(t)\frac{d^2}{dt^2} + p_1(t)\frac{d}{dt} + p_2(t)\right) S(t) = p_3(t)$$

p_0, p_1, p_2, p_3 are polynomials in t (of degrees 7, 6, 5, 4) and in m_1^2, m_2^2, m_3^2.

Ansatz for the solution:

$$S(t) = C_1\psi_1(t) + C_2\psi_2(t) + \int_0^t dt_1 \frac{p_3(t_1)}{p_0(t_1)W(t_1)} \left(-\psi_1(t)\psi_2(t_1) + \psi_2(t)\psi_1(t_1)\right)$$

with the solutions of the homogeneous equation ψ_1, ψ_2, constants C_1, C_2,

Wronski determinant $W(t) = \psi_1(t)\frac{d}{dt}\psi_2(t) - \psi_2(t)\frac{d}{dt}\psi_1(t)$

We will use

- complete elliptic integral of the first kind:

$$K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

- complete elliptic integral of the second kind:

$$E(k) = \int_0^1 \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}dx$$

- functions $k(t)$, $k'(t)$ such that $k(t)^2 + k'(t)^2 = 1$

Introduce the notation

$$x_1 = (m_1 - m_2)^2, \; x_2 = (m_3 - \sqrt{t})^2, \; x_3 = (m_3 + \sqrt{t})^2, \; x_4 = (m_1 + m_2)^2$$

Consider the **auxiliary elliptic curve** given by the equation

$$y^2 = (x - x_1)(x - x_2)(x - x_3)(x - x_4).$$

By the associated holomorphic 1-form dx/y one obtains the period integrals

$$\psi_1(t) = 2 \int_{x_2}^{x_3} \frac{dx}{y} = \frac{4}{\xi(t)} K\left(k(t)\right),$$

$$\psi_2(t) = 2 \int_{x_4}^{x_3} \frac{dx}{y} = \frac{4i}{\xi(t)} K\left(k'(t)\right)$$

$$\text{with } \xi(t) = \sqrt{(x_3 - x_1)(x_4 - x_2)},$$
$$k(t) = \sqrt{\tfrac{(x_3-x_2)(x_4-x_1)}{(x_3-x_1)(x_4-x_2)}}, \; k'(t) = \sqrt{\tfrac{(x_2-x_1)(x_4-x_3)}{(x_3-x_1)(x_4-x_2)}}, \; k(t)^2 + k'(t)^2 = 1$$

$\psi_1(t)$ and $\psi_2(t)$ **solve the homogeneous differential equation** for $S(t)$.

Furthermore, from integrating over $\frac{xdx}{y}$ we obtain

$$\phi_1(t) = \frac{4}{\xi(t)} \left(K(k(t)) - E\left(k(t)\right) \right)$$

$$\phi_2(t) = \frac{4i}{\xi(t)} E\left(k'(t)\right)$$

The period matrix of the elliptic curve is

$$\begin{pmatrix} \psi_1(t) & \psi_2(t) \\ \phi_1(t) & \phi_2(t) \end{pmatrix}$$

and we have the Legendre relation

$$\psi_1(t)\phi_2(t) - \psi_2(t)\phi_1(t) = \frac{8\pi i}{\xi(t)}.$$

These are appropriate functions to express the full solution.

Full solution:

$$S(t) = \frac{1}{\pi}\left(\sum_{i=1}^{3} \mathrm{Cl}_2(\alpha_i)\right)\psi_1(t) + \frac{1}{i\pi}\int_0^t dt_1 \left(\eta_1(t_1) - \frac{b_1 t_1 - b_0}{3(x_2 - x_1)(x_4 - x_3)}\left(\eta_2(t_1) - \eta_1(t_1)\right)\right)$$

where

$$\eta_1(t_1) = \psi_2(t)\psi_1(t_1) - \psi_1(t)\psi_2(t_1)$$

$$\eta_2(t_1) = \psi_2(t)\phi_1(t_1) - \psi_1(t)\phi_2(t_1)$$

Clausen function: $\mathrm{Cl}_2(x) = \frac{1}{2i}\left(\mathrm{Li}_2(e^{ix}) - \mathrm{Li}(e^{-ix})\right)$
$\alpha_i = 2\mathrm{arctan}\left(\frac{\sqrt{\Delta}}{\delta_i}\right)$, Δ, δ_i : polynomials in m_1, m_2, m_3 of degrees 4 and 2 resp.

$b_i = d_i(m_1, m_2, m_3)\ln(m_1^2) + d_i(m_2, m_3, m_1)\ln(m_2^2) + d_i(m_3, m_1, m_2)\ln(m_3^2)$,
$d_1(m_1, m_2, m_3) = 2m_1^2 - m_2^2 - m_3^2$,
$d_0(m_1, m_2, m_3) = 2m_1^4 - m_2^4 - m_3^4 - m_1^2 m_2^2 - m_1^2 m_3^2 + 2m_2^2 m_3^2$

Conclusions:

- Multiple polylogarithms in several variables are homotopy invariant iterated integrals with particularly good properties. We want to use them to iteratively integrate out Feynman parameters.

- To decide whether the approach can succeed there is a criterion of linear reducibility on the graphs. The class of linearly reducible graphs is minor-closed. This allows for a convenient classification by forbidden minors.

- We obtained a new result for the sunrise integral with arbitrary masses. The result contains integrals over elliptic integrals and can be built up from the period integrals of an (auxiliary) elliptic curve. An expression in terms of iterated integrals would probably require further extensions of polylogarithms.

A well known **functional equation** is the five-term-relation:

$$-\mathrm{Li}_2\left(\frac{1-y}{1-\frac{1}{x}}\right)-\mathrm{Li}_2\left(\frac{1-x}{1-\frac{1}{y}}\right)+\mathrm{Li}_2(xy)-\mathrm{Li}_2(x)-\mathrm{Li}_2(y)=\frac{1}{2}\ln^2(1-x)+\frac{1}{2}\ln^2(1-y)$$

Writing each function as iterated integral on the total space (using ψ), the relation becomes obvious:

$$\mathrm{Li}_2\left(\frac{1-y}{1-\frac{1}{x}}\right)=\left[\frac{dx}{x}+\frac{dx}{1-x}-\frac{dy}{1-y}\middle|\frac{xdy+ydx}{1-xy}\right]-\left[\frac{dx}{1-x}\middle|\frac{dy}{1-y}\right]-\left[\frac{dx}{x}+\frac{dx}{1-x}\middle|\frac{dx}{1-x}\right]$$

$$\mathrm{Li}_2\left(\frac{1-x}{1-\frac{1}{y}}\right)=\left[\frac{dy}{y}+\frac{dy}{1-y}-\frac{dx}{1-x}\middle|\frac{xdy+ydx}{1-xy}\right]+\left[\frac{dx}{1-x}\middle|\frac{dy}{1-y}\right]-\left[\frac{dy}{y}+\frac{dy}{1-y}\middle|\frac{dy}{1-y}\right]$$

$$\mathrm{Li}_2(xy)=\left[\frac{dx}{x}+\frac{dy}{y}\middle|\frac{xdy+ydx}{1-xy}\right],\ \mathrm{Li}_2(x)=\left[\frac{dx}{x}\middle|\frac{dx}{1-x}\right],\ \mathrm{Li}_2(y)=\left[\frac{dy}{y}\middle|\frac{dy}{1-y}\right]$$

- If the polynomials $S = \{f_1, \ldots, f_n\}$ are linear in a Feynman parameter x_{r_1}, consider:

$$f_i = g_i x_{r_1} + h_i, \; g_i = \frac{\partial f_i}{x_{r_1}}, \; h_i = f_i|_{x_{r_1}=0}$$

- $S_{(r_1)}$ = irreducible factors of $\{g_i\}_{1 \le i \le n}, \{h_i\}_{1 \le i \le n}, \{h_i g_j - g_i h_j\}_{1 \le i < j \le n}$
- iterate for a sequence $(x_{r_1}, x_{r_2}, \ldots, x_{r_n}) \Rightarrow S_{(r_1)}, S_{(r_1, r_2)}, \ldots, S_{(r_1, \ldots, r_n)}$
- take intersections:

$$\begin{aligned} S_{[r_1, r_2]} &= S_{(r_1, r_2)} \cap S_{(r_2, r_1)} \\ S_{[r_1, r_2, \ldots, r_k]} &= \cap_{1 \le i \le k} S_{[r_1, \ldots, \hat{r}_i, \ldots, r_k](r_i)}, \; k > 3 \end{aligned}$$

$$x_{r_1}, x_{r_2}, \ldots, x_{r_n} \Rightarrow S_{(r_1)}, S_{[r_1, r_2]}, \ldots, S_{[r_1, \ldots, r_n]}$$

Def.: A set S is linearly reducible if there is an ordering $(x_{r_1}, x_{r_2}, \ldots, x_{r_n})$ such that for all $1 \le k \le n$ every polynomial in $S_{[r_1, \ldots, r_k]}$ is linear in $x_{r_{k+1}}$.

If this is true for $\{\mathcal{U}_G, \mathcal{F}_G\}$ we say that the Feynman graph G is linearly reducible.