Scale Invariance and Conformal Invariance

Jean-François Fortin

Theory Division, Department of Physics, CERN, CH-1211 Geneva 23, Switzerland
SITP, Department of Physics, Stanford University, Stanford, CA 94305, USA

May 13, 2013
Johannes Gutenberg Universität Mainz

based on

arXiv:1106.2540, 1107.3840, 1110.1634, 1202.4757, 1206.2921, 1208.3674, 1210.2718 [hep-th] and work in progress

with Benjamín Grinstein, Christopher W. Murphy and Andreas Stergiou

Quiz

Questions about scale invariance and conformal conformal invariance

- Do scale field theories (SFTs) live at RG flow fixed points ?
- Do conformal field theories (CFTs) live at RG flow fixed points ?
- Does scale invariance imply conformal invariance ?
  - Proved to be true for $d = 2$ unitarity interacting quantum field theories (QFTs) with well-defined correlation functions Zamolodchikov (1986) & Polchinski (1988)
  - Assumed to be true for $d = 4$

Question about possible types of RG flows

- Are there RG flow recurrent behaviors ?

Quiz answers

Questions about scale invariance and conformal conformal invariance

- ✗ Do scale field theories (SFTs) live at RG flow fixed points ?
- ✗ Do conformal field theories (CFTs) live at RG flow fixed points ?
- ✓ Does scale invariance imply conformal invariance ?
  - Proved to be true for $d = 2$ unitarity interacting quantum field theories (QFTs) with well-defined correlation functions Zamolodchikov (1986) & Polchinski (1988)
  - Assumed to be true for $d = 4$

Question about possible types of RG flows

- ✓ Are there RG flow recurrent behaviors ?

Why study these questions ?

QFT phases

- Infrared (IR) free
  - With mass gap ⇒ Exponentially-decaying correlation functions (e.g. Higgs phase)
  - Without mass gap ⇒ Trivial power-law correlation functions (e.g. Abelian Coulomb phase)
- IR interacting
  - CFTs ⇒ Power-law correlation functions (e.g. non-Abelian Coulomb phase)
  - SFTs ⇒ ?

Possible types of RG flows

- Strong coupling
- Weak coupling
  - Fixed points (e.g. Banks-Zaks fixed point Banks, Zaks (1982))
  - Recurrent behaviors (e.g. limit cycles or ergodic behaviors)

Outline

Preliminaries ($d > 2$)

- Dilatation current Wess (1960)
  - $\mathcal{D}^\mu(x) = x^\nu T_\nu{}^\mu(x) - V^\mu(x)$
  - $T_\nu{}^\mu(x)$ any symmetric EM tensor following from spacetime nature of scale transformations
  - $V^\mu(x)$ local operator (virial current) contributing to scale dimensions of fields
  - Freedom in choice of $T_\nu{}^\mu(x)$ compensated by freedom in choice of $V^\mu(x)$
- Scale invariance $\Rightarrow T_\mu{}^\mu(x) = \partial_\mu V^\mu(x)$

- Conformal current Wess (1960)
  - $\mathcal{C}_{\nu}{}^{\mu}(x) = v_{\nu}{}^{\lambda}(x) T_{\lambda}{}^{\mu}(x) - (\partial_{\lambda} v_{\nu}{}^{\lambda})(x) V'^{\mu}(x) + (\partial_{\rho}\partial_{\lambda} v_{\nu}{}^{\lambda})(x) L^{\rho\mu}(x)$
  - $T_{\lambda}{}^{\mu}(x)$ any symmetric EM tensor following from spacetime nature of conformal transformations
  - $V'^{\mu}(x)$ local operator corresponding to ambiguity in choice of dilatation current
  - $L^{\rho\mu}(x)$ local symmetric operator correcting position dependence of scale factor
  - $(\partial_{\lambda} v_{\nu}{}^{\lambda})(x)$ scale factor (general linear function of x_{ν})
  - Freedom in choice of $T_{\lambda}{}^{\mu}(x)$ compensated by freedom in choice of $V'^{\mu}(x)$ and $L^{\rho\mu}(x)$
- Conformal invariance $\Rightarrow$ $T_{\mu}{}^{\mu}(x) = \partial_{\mu} V'^{\mu}(x) = \partial_{\mu}\partial_{\nu} L^{\nu\mu}(x)$
- Conformal invariance $\Rightarrow$ Existence of symmetric traceless energy-momentum tensor Polchinski (1988)

Scale without conformal invariance

Non-conformal scale-invariant QFTs Polchinski (1988)

- Scale invariance $\Rightarrow$ $T_\mu{}^\mu(x) = \partial_\mu V^\mu(x)$
- Conformal invariance $\Rightarrow$ $T_\mu{}^\mu(x) = \partial_\mu \partial_\nu L^{\nu\mu}(x)$
- Scale without conformal invariance

  $\Rightarrow$ $T_\mu{}^\mu(x) = \partial_\mu V^\mu(x)$ where $V^\mu(x) \neq J^\mu(x) + \partial_\nu L^{\nu\mu}(x)$ with $\partial_\mu J^\mu(x) = 0$
- Constraints on possible virial current candidates
  - Gauge invariant (spatial integral)
  - Fixed $d-1$ scale dimension in d spacetime dimensions
- No suitable virial current $\Rightarrow$ Scale invariance implies conformal invariance (examples: ϕ^p in $d = n - \epsilon$ for $(p, n) = (6, 3), (4, 4)$ and $(3, 6)$)

Virial current candidates

Most general classically scale-invariant renormalizable theory in $d = 4 - \epsilon$ spacetime dimensions Jack, Osborn (1985)

$$
\begin{aligned}
\mathcal{L} = & -\mu^{-\epsilon} Z_A \frac{1}{4g_A^2} F^A_{\mu\nu} F^{A\mu\nu} + \frac{1}{2} Z^{\frac{1}{2}}_{ab} Z^{\frac{1}{2}}_{ac} D_\mu \phi_b D^\mu \phi_c \\
& + \frac{1}{2} Z^{\frac{1}{2}*}_{ij} Z^{\frac{1}{2}}_{ik} \bar{\psi}_j i \bar{\sigma}^\mu D_\mu \psi_k - \frac{1}{2} Z^{\frac{1}{2}*}_{ij} Z^{\frac{1}{2}}_{ik} D_\mu \bar{\psi}_j i \bar{\sigma}^\mu \psi_k \\
& - \frac{1}{4!} \mu^{\epsilon} (\lambda Z^\lambda)_{abcd} \phi_a \phi_b \phi_c \phi_d \\
& - \frac{1}{2} \mu^{\frac{\epsilon}{2}} (y Z^y)_{a|ij} \phi_a \psi_i \psi_j - \frac{1}{2} \mu^{\frac{\epsilon}{2}} (y Z^y)^*_{a|ij} \phi_a \bar{\psi}_i \bar{\psi}_j
\end{aligned}
$$

- $\phi_a(x)$ real scalar fields
- $\psi_i^\alpha(x)$ Weyl fermions
- $A_\mu^A(x)$ gauge fields
- Dimensional regularization with minimal subtraction

Virial current candidates and new improved EM tensor

- Virial current $V^\mu(x) = Q_{ab}\phi_a D^\mu \phi_b - P_{ij}\bar{\psi}_i i\bar{\sigma}^\mu \psi_j$
  - $Q_{ba} = -Q_{ab}$
  - $P^*_{ji} = -P_{ij}$
- New improved energy-momentum tensor $[\Theta_\nu{}^\mu(x)]$ Callan, Coleman, Jackiw (1970)
  - Finite and not renormalized (vanishing anomalous dimension)
  - Anomalous trace Osborn (1989,1991) & Jack, Osborn (1990)

$$[\Theta_\mu{}^\mu(x)] = \frac{\mathrm{B}_A}{2g_A^3}[F^A_{\mu\nu}F^{A\mu\nu}] - \frac{1}{4!}\mathrm{B}_{abcd}[\phi_a\phi_b\phi_c\phi_d] - \frac{1}{2}\left(\mathrm{B}_{a|ij}[\phi_a\psi_i\psi_j] + \text{h.c.}\right) - ((\delta+\Gamma)f)\cdot\frac{\delta}{\delta f}\mathcal{A}$$

- Anomalous trace

$$[\Theta_\mu{}^\mu(x)] = \mathrm{B}^I[\mathcal{O}_I(x)] - ((\delta + \Gamma)f) \cdot \frac{\delta}{\delta f}A$$

- Conserved dilatation current $\partial_\mu \mathcal{D}^\mu(x) = 0$ (up to EOMs)

$$\mathrm{B}^I = \mathcal{Q}^I \equiv -(gQ)^I$$

- Conserved conformal current $\partial_\mu \mathcal{C}_\nu{}^\mu(x) = 0$ (up to EOMs)

$$\mathrm{B}^I = 0$$

Virial current and unitarity bounds

- New improved energy-momentum tensor ⇒ Finite and not renormalized Callan, Coleman, Jackiw (1970)
- Operators related to EOMs ⇒ Finite and not renormalized Politzer (1980) & Robertson (1991)
- Virial current ⇒ Finite and not renormalized
  - Unconserved current with scale dimension exactly 3
- Unitarity bounds for conformal versus scale-invariant QFTs Grinstein, Intriligator, Rothstein (2008)
- Non-trivial virial current ⇒ Non-conformal scale-invariant QFTs

RG flows along scale-invariant trajectories

Scale-invariant solution $(\lambda_{abcd}, y_{a|ij}, g_A) \Rightarrow$ RG trajectory

$$
\begin{aligned}
\bar{\lambda}_{abcd}(t) &= \widehat{Z}_{a'a}(t)\widehat{Z}_{b'b}(t)\widehat{Z}_{c'c}(t)\widehat{Z}_{d'd}(t)\lambda_{a'b'c'd'} \\
\bar{y}_{a|ij}(t) &= \widehat{Z}_{a'a}(t)\widehat{Z}_{i'i}(t)\widehat{Z}_{j'j}(t)y_{a'|i'j'} \\
\bar{g}_A(t) &= g_A
\end{aligned}
$$

$$
\left.\begin{aligned}
\widehat{Z}_{a'a}(t) &= \left(e^{Qt}\right)_{a'a} \\
\widehat{Z}_{i'i}(t) &= \left(e^{Pt}\right)_{i'i}
\end{aligned}\right\} \quad t = \ln(\mu_0/\mu) \quad \text{(RG time)}
$$

- $(\bar{\lambda}_{abcd}(t,g,\lambda,y), \bar{y}_{a|ij}(t,g,\lambda,y), \bar{g}_A(t,g,\lambda,y))$ also scale-invariant solution
- Q_{ab} and P_{ij} constant along RG trajectory
- $\widehat{Z}_{ab}(t)$ orthogonal and $\widehat{Z}_{ij}(t)$ unitary $\Rightarrow$ Always non-vanishing beta-functions along scale-invariant trajectory

Scale invariance and recurrent behaviors

RG flows along scale-invariant trajectories $\Rightarrow$ Recurrent behaviors !

Lorenz (1963,1964), Wilson (1971) & Kogut, Wilson (1974)

- Virial current $\Rightarrow$ Transformation in symmetry group of kinetic terms ($SO(N_S) \times U(N_F)$)
  - $\widehat{Z}_{ab}(t)$ and $\widehat{Z}_{ij}(t)$ in $SO(N_S) \times U(N_F)$
  - Q_{ab} antisymmetric and P_{ij} antihermitian $\Rightarrow$ Purely imaginary eigenvalues

$\Rightarrow$ Periodic (limit cycle) or quasi-periodic (ergodicity) scale-invariant trajectories

Recurrent behaviors

Intuition from $\mathcal{D}^{\mu}(x) = x^{\nu} T_{\nu}{}^{\mu}(x) - V^{\mu}(x)$

- RG flow $\Rightarrow$ Generated by scale transformation $(x^{\nu} T_{\nu}{}^{\mu}(x))$
- RG flow $\Rightarrow$ Related to virial current through conservation of dilatation current
- Virial current $\Rightarrow$ Generates internal transformation of the fields
  - Internal transformation in compact group $SO(N_S) \times U(N_F)$

  $\Rightarrow$ Rotate back to or close to identity
- RG flow return back to or close to identity $\Rightarrow$ Recurrent behavior

Scale-invariant trajectories ?

RG flows $\sim$ Field redefinitions $\Rightarrow$ Scale-invariant trajectories or fixed points ?

- RG-time-dependent field redefinitions $\Rightarrow$ Generates RG flows
  Wegner (1974) & Latorre, Morris (2001)
  - RG-time-dependent field redefinitions $\Rightarrow$ All exact RG flows (Wilson, Wegner, Polchinski, etc.)

Beta-function operators $\sim$ Redundant operators $\Rightarrow$ Scale-invariant trajectories or fixed points ?

- Wavefunction renormalization operators $\Rightarrow$ Redundant operators
  - Redundant beta-function operators necessary for scale invariance

Scale-invariant QFTs $\Rightarrow$ Non-trivial RG flows (recurrent behaviors)

Why dilatation generators generate dilatations

Dilatation generators do not generate dilatations in non-scale-invariant QFTs Coleman, Jackiw (1971)

- Quantum anomalies at low orders
  - Anomalous dimensions

  ⇒ Possible to absorb into redefinition of scale dimensions of fields

  ✔ Preserve scale invariance

- Quantum anomalies at high orders
  - Beta-functions

  ⇒ Not possible to absorb

  ✗ Break scale invariance

Why dilatation generators generate dilatations in scale-invariant QFTs ?

- Beta-functions on scale-invariant trajectories
  - Both vertex correction and wavefunction renormalization contributions
  - Very specific form for vertex correction contribution
  - Equivalent in form to wavefunction renormalization contribution (redundant operators)

  $\Rightarrow$ Also possible to absorb into redefinition of scale dimensions of fields

  ✓ Preserve scale invariance !

- Beta-functions from vertex corrections and wavefunction renormalizations ($d = 4$ spacetime dimensions)

$$
\begin{aligned}
\mathrm{B}_{abcd} &= -\frac{d\lambda_{abcd}}{dt} \\
&= -(\lambda\gamma^{\lambda})_{abcd} + \lambda_{a'bcd}\Gamma_{a'a} + \lambda_{ab'cd}\Gamma_{b'b} + \lambda_{abc'd}\Gamma_{c'c} + \lambda_{abcd'}\Gamma_{d'd} \\
\mathrm{B}_{a|ij} &= -\frac{dy_{a|ij}}{dt} = -(y\gamma^{y})_{a|ij} + y_{a'|ij}\Gamma_{a'a} + y_{a|i'j}\Gamma_{i'i} + y_{a|ij'}\Gamma_{j'j} \\
\mathrm{B}_{A} &= -\frac{dg_A}{dt} = \gamma_A g_A \quad \text{(no sum)}
\end{aligned}
$$

- Beta-functions on scale-invariant trajectories

$$
\begin{aligned}
\mathrm{B}_{abcd} &= -\lambda_{a'bcd}Q_{a'a} - \lambda_{ab'cd}Q_{b'b} - \lambda_{abc'd}Q_{c'c} - \lambda_{abcd'}Q_{d'd} \\
\mathrm{B}_{a|ij} &= -y_{a'|ij}Q_{a'a} - y_{a|i'j}P_{i'i} - y_{a|ij'}P_{j'j} \\
\mathrm{B}_{A} &= 0
\end{aligned}
$$

Ward identity for scale invariance

Callan-Symanzik equation for effective action Callan (1970) & Symanzik (1970)

$$\left[M\frac{\partial}{\partial M}+B^I\frac{\partial}{\partial g^I}+\Gamma_J^{\;I}\int d^4x\, f_I(x)\frac{\delta}{\delta f_J(x)}\right]\Gamma[f(x),g,M]=0$$

- In non-scale-invariant QFTs
  - ✔ Anomalous dimensions
  - ✗ Beta-functions

$$\left[M\frac{\partial}{\partial M}+(\Gamma+Q)_J^{\;I}\int d^4x\, f_I(x)\frac{\delta}{\delta f_J(x)}\right]\Gamma[f(x),g,M]=0$$

- In scale-invariant QFTs
  - ✔ Anomalous dimensions
  - ✔ Beta-functions (redundant operators)

Poincaré algebra augmented with dilatation charge

- Beta-functions on scale-invariant trajectories
  - Quantum-mechanical generation of scale dimensions
  - Appropriate scale dimensions required by virial current
  - $\Rightarrow$ Conserved dilatation current $\mathcal{D}^\mu(x)$

- Poincaré algebra with dilatation charge $D = \int d^3x\, \mathcal{D}^0(x)$

$$
\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho}) \\
[M_{\mu\nu}, P_\rho] &= -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \\
[D, P_\mu] &= -iP_\mu
\end{aligned}
$$

- Algebra action on fields $\mathcal{O}_I(x)$

$$
\begin{aligned}
[M_{\mu\nu}, \mathcal{O}_I(x)] &= -i(x_\mu\partial_\nu - x_\nu\partial_\mu + \Sigma_{\mu\nu})\mathcal{O}_I(x) \\
[P_\mu, \mathcal{O}_I(x)] &= -i\partial_\mu\mathcal{O}_I(x) \\
[D, \mathcal{O}_I(x)] &= -i(x\cdot\partial + \Delta)\mathcal{O}_I(x)
\end{aligned}
$$

- New classical scale dimensions of fields due to virial current

$$[D, \phi_a(x)] = -i(x \cdot \partial + 1)\phi_a(x) - iQ_{ab}\phi_b(x)$$
$$[D, \psi_i(x)] = -i(x \cdot \partial + \tfrac{3}{2})\psi_i(x) - iP_{ij}\psi_j(x)$$

  - How do non-conformal scale-invariant QFTs know about new scale dimensions ?

  $\Rightarrow$ Generated by beta-functions !

- Quantum-mechanical scale dimensions of fields

$$\Delta_{ab} = \delta_{ab} + \Gamma_{ab} + Q_{ab}$$
$$\Delta_{ij} = \tfrac{3}{2}\delta_{ij} + \Gamma_{ij} + P_{ij}$$

Scale-invariant trajectories ??

Beta-functions $\sim$ Anomalous dimensions $\Rightarrow$ Scale-invariant trajectories or fixed points ?

- Shift beta-functions away $\Rightarrow$ Scheme change
  - ✗ RG flow reccurent behaviors $\Rightarrow$ RG flow fixed points

Non-conformal scale-invariant QFTs $\Rightarrow$ Non-trivial RG flows

Scheme changes and RG flow fixed points

One-coupling case Gross (1975)

$$g \to \tilde{g}(g) = g + \mathcal{O}(g^3)$$

$$Z^{1/2}(g) \to \widetilde{Z}^{1/2}(\tilde{g}) = Z^{1/2}(g)F(g) \quad \text{where} \quad F(g) = 1 + \mathcal{O}(g^2)$$

$$\mathrm{B}(g) = -\frac{dg}{dt} \qquad \tilde{\mathrm{B}}(\tilde{g}) = \mathrm{B}(g)\frac{\partial \tilde{g}}{\partial g}$$

$$\Gamma(g) = -Z^{-1/2}(g)\frac{dZ^{1/2}(g)}{dt} \qquad \tilde{\Gamma}(\tilde{g}) = \Gamma(g) + F^{-1}(g)\,\mathrm{B}(g)\frac{\partial F(g)}{\partial g}$$

- Scheme-independent properties
  - Existence of RG flow fixed point $\mathrm{B}(g_*) = 0$
  - $\Gamma(g_*) \Rightarrow$ Scaling behavior of Green functions
  - $\partial\,\mathrm{B}(g)/\partial g|_{g=g_*} \Rightarrow$ Character of RG flow fixed point
  - First two coefficients in $\mathrm{B}(g) \Rightarrow$ UV or IR coupling asymptotics (remaining terms can all be set to vanish)
  - First coefficient in $\Gamma(g) \Rightarrow$ UV or IR field scale factor

Scheme changes and scale-invariant trajectories

Multi-coupling case JFF, Grinstein, Stergiou (2012)

$$\tilde{\mathrm{B}}(\tilde{g}) = -\frac{d\tilde{g}}{dt} = \mathrm{B}(g)\frac{\partial \tilde{g}}{\partial g}$$

$$\tilde{\Gamma}(\tilde{g}) = F^{-1}(g)\Gamma(g)F(g) + F^{-1}(g)\,\mathrm{B}(g)\frac{\partial F(g)}{\partial g}$$

$$-\frac{df(g)}{dt} = -f(g)Q \quad \text{(on SFTs)}$$

- (Natural) scheme-independent properties
  - Existence of scale-invariant trajectory $\mathrm{B}(g_*) = -gQ$
  - $\Gamma(g_*) + Q$ eigenvalues $\Rightarrow$ Scaling behavior of Green functions
  - $\partial\,\mathrm{B}(g)/\partial g|_{g=g_*} + Q$ eigenvalues $\Rightarrow$ Character of scale-invariant trajectory
  - First coefficient in $\mathrm{B}(g)$ $\Rightarrow$ UV or IR coupling asymptotics (remaining terms cannot all be set to vanish)
  - First coefficient in $\Gamma(g)$ $\Rightarrow$ UV or IR field scale factor

Non-conformal scale-invariant correlation functions

- Scalar fields $\mathcal{O}_I(x)$ with scale dimensions Δ_I

$$\langle \mathcal{O}_I(x_1)\mathcal{O}_J(x_2)\rangle = \frac{g_{IJ}}{(x_1-x_2)^{\Delta_I+\Delta_J}}$$

$$\langle \mathcal{O}_I(x_1)\mathcal{O}_J(x_2)\mathcal{O}_K(x_3)\rangle = \sum_{\substack{\delta_1+\delta_2+\delta_3=\\ \Delta_I+\Delta_J+\Delta_K}} \frac{c_{IJK}^{\delta_1\delta_2\delta_3}}{(x_1-x_2)^{\delta_1}(x_2-x_3)^{\delta_2}(x_3-x_1)^{\delta_3}}$$

  - Non-vanishing two-point functions with $\Delta_I \neq \Delta_J$ contrary to CFTs

- Two-point correlation functions of fundamental real scalar fields

$$\langle \phi_a(x)\phi_b(0)\rangle = \left[(x^2)^{-\frac{\Delta}{2}}\, G^{\phi}\, (x^2)^{-\frac{\Delta^T}{2}}\right]_{ab}$$

  - G^{ϕ} constant real symmetric matrix

- Two-point correlation functions of scalar operators $\mathcal{O}_a(x)$

$$\langle \mathcal{O}_a(x)\mathcal{O}_b(0)\rangle = \left[(x^2)^{-\frac{\Delta}{2}}\, G\, (x^2)^{-\frac{\Delta^T}{2}}\right]_{ab}$$
$$= i\int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot x}\left[(-p^2-i\epsilon)^{\frac{\Delta}{2}-1}\,\widetilde{G}\,(-p^2-i\epsilon)^{\frac{\Delta^T}{2}-1}\right]_{ab}$$

  - G (and $\widetilde{G}$) constant real symmetric matrices

- Two-point correlation functions of vector operators $\mathcal{O}_a^\mu(x)$

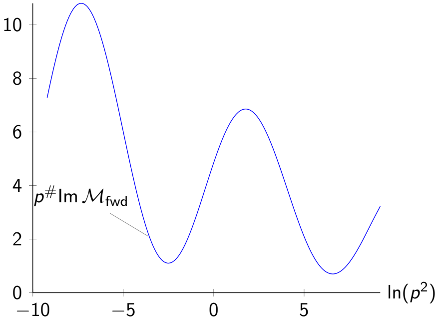
$$\langle \mathcal{O}_a^\mu(x)\mathcal{O}_b^\nu(0)\rangle = \left[(x^2)^{-\frac{\Delta}{2}}\left(g^{\mu\nu}A+\frac{x^\mu x^\nu}{x^2}B\right)(x^2)^{-\frac{\Delta^T}{2}}\right]_{ab}$$
$$= -i\int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot x}\left[(-p^2-i\epsilon)^{\frac{\Delta}{2}-1}\left(g^{\mu\nu}\widetilde{A}+\frac{p^\mu p^\nu}{p^2}\widetilde{B}\right)(-p^2-i\epsilon)^{\frac{\Delta^T}{2}-1}\right]_{ab}$$

  - A and B (and $\widetilde{A}$ and $\widetilde{B}$) constant real symmetric matrices

Coupled QFT/SFT where $\mathcal{L} \supset g_a \chi \mathcal{O}_a + \text{h.c.}$ with external source χ and scalar operator $\mathcal{O}_a$

$$\mathcal{M} = g_a g_b |\chi|^2 \left[(-p^2 - i\epsilon)^{\frac{\Delta}{2}-1} \, \widetilde{G} \, (-p^2 - i\epsilon)^{\frac{\Delta^T}{2}-1} \right]_{ab}$$

$$\mathrm{Im}\,\mathcal{M}_{\text{fwd}} = g_a g_b |\chi|^2 \left[(p^2)^{\frac{\Delta}{2}-1} \left\{ \cos\left[\left(1-\tfrac{\Delta}{2}\right)\pi\right] \widetilde{G} \sin\left[\left(1-\frac{\Delta^T}{2}\right)\pi\right] + \sin\left[\left(1-\tfrac{\Delta}{2}\right)\pi\right] \widetilde{G} \cos\left[\left(1-\frac{\Delta^T}{2}\right)\pi\right] \right\} (p^2)^{\frac{\Delta^T}{2}-1} \right]_{ab} \theta(p^0)\theta(p^2)$$



Stability properties

Character of scale-invariant trajectory JFF, Grinstein, Stergiou (2012)

$$\delta g(t) = [g(t) - g_*(t)]e^{-Qt} = g(t)e^{-Qt} - g_*(0)$$

$$-\frac{d\,\delta g(t)}{dt} = [\mathrm{B}(t) - \mathcal{Q}(t)]e^{-Qt} + \delta g(t) Q = \delta g(t) S + \cdots$$

$$\mathrm{B}(t) = \mathrm{B}|_{g=g_*(t)} + [g(t) - g_*(t)]\,\frac{\partial\,\mathrm{B}(g)}{\partial g}\bigg|_{g=g_*(t)} + \cdots$$

$$= \mathcal{Q}(t) + \delta g(t)\,\frac{\partial\,\mathrm{B}(g)}{\partial g}\bigg|_{g=g_*(0)} e^{Qt} + \cdots$$

- Deformations $\delta g(t) = \delta g(0) e^{-St} + \cdots$
  - Behavior of deformations in "comoving frame"
  - Choice of $g_*(0)$ arbitrary
- Stability matrix $S = \partial\,\mathrm{B}(g)/\partial g|_{g=g_*(0)} + Q$
  - Scheme-independent eigenvalues
  - Positive (negative) eigenvalues $\Rightarrow$ IR attractive (repulsive) deformations
  - Special eigenvector $\delta g(0) \propto \mathcal{Q}(0)$ with vanishing eigenvalue $\Rightarrow$ Deformation along scale-invariant trajectory

Scale invariance, c-theorem and gradient flows

c-theorem Barnes, Intriligator, Wecht, Wright (2004)

- RG flow $\Rightarrow$ Irreversible process (integrating out DOFs)
- $c(g) \sim$ measure of number of massless DOFs
- c-theorem
  - weak ($c_{IR} < c_{UV}$) Komargodski, Schwimmer (2011) & Luty, Polchinski, Rattazzi (2012)
  - stronger ($\frac{dc}{dt} \leq 0$) Osborn (1989,1991) & Jack, Osborn (1990)
  - ~~strongest~~ (RG flows as gradient flows)

- Gradient flow

$$\mathrm{B}^I(g) = -\frac{dg^I}{dt} = G^{IJ}(g)\frac{\partial c(g)}{\partial g^J}$$

  - G^{IJ} positive-definite metric
  - Potential $c(g)$ function of couplings

- Potential $c(g)$ monotonically decreasing along RG trajectory

$$\frac{dc(g(t))}{dt} = -G_{IJ}(g)\,\mathrm{B}^I\,\mathrm{B}^J \leq 0$$

  - Recurrent behaviors (scale-invariant trajectories) $\nLeftrightarrow$ Gradient flows (scale implies conformal invariance) Wallace, Zia (1975)

Polchinski–Dorigoni–Rychkov argument at one loop

Non-conformal scale-invariant beta-functions

$$B_{abcd} = \mathcal{Q}_{abcd} \equiv -Q_{a'a}\lambda_{a'bcd} - Q_{b'b}\lambda_{ab'cd} - Q_{c'c}\lambda_{abc'd} - Q_{d'd}\lambda_{abcd'}$$
$$B_{a|ij} = \mathcal{P}_{a|ij} \equiv -Q_{a'a}y_{a'|ij} - P_{i'i}y_{a|i'j} - P_{j'j}y_{a|ij'}$$

- Real scalar fields only Polchinski (1988)
  - $\mathcal{Q}_{abcd}\, B^{(\text{one-loop})}_{abcd} = 0 \Rightarrow \mathcal{Q}_{abcd} = 0$

  $\Rightarrow$ Scale invariance implies conformal invariance

- Real scalar fields and Weyl fermions Dorigoni, Rychkov (2009)
  - $\mathcal{P}^{*}_{a|ij}\, B^{(\text{one-loop})}_{a|ij} = 0 \Rightarrow \mathcal{P}_{a|ij} = 0$
  - $\mathcal{Q}_{abcd}\, B^{(\text{one-loop})}_{abcd} = 0$ using $\mathcal{P}_{a|ij} = 0 \Rightarrow \mathcal{Q}_{abcd} = 0$

  $\Rightarrow$ Scale invariance implies conformal invariance

Polchinski–Dorigoni–Rychkov argument at two loops

- Real scalar fields only JFF, Grinstein, Stergiou (2011)
  - $\mathcal{Q}_{abcd}\,\mathrm{B}^{(\text{two-loop})}_{abcd}=0\Rightarrow\mathcal{Q}_{abcd}=0$

  $\Rightarrow$ Scale invariance implies conformal invariance

- One real scalar field only and Weyl fermions *ibid*
  - $\mathcal{P}^*_{a|ij}\,\mathrm{B}^{(\text{two-loop})}_{a|ij}=0\Rightarrow\mathcal{P}_{a|ij}=0$
  - $\mathcal{Q}_{abcd}\equiv 0$

  $\Rightarrow$ Scale invariance implies conformal invariance (also at all loops)

- Real scalar fields and Weyl fermions *ibid*
  - $\mathcal{P}^*_{a|ij}\,\mathrm{B}^{(\text{two-loop})}_{a|ij}\neq 0$
  - Scale invariance does NOT imply conformal invariance
  - Obstruction due to $y^3\lambda$ and $y\lambda^2$ terms (also obstruction to gradient flow interpretation Wallace, Zia (1975))

  $\Rightarrow$ Obstruction nevertheless allows for weakly-coupled gradient flow interpretation Jack, Osborn (1990)

Interference between $B_a^{\text{(two-loop)}}$ and $B_{abcd}^{\text{(one-loop)}}$

$$c \supset d_1 \text{tr}(y_a^* y_b y_c^* y_d)\lambda_{abcd} + d_2 \text{tr}(y_a^* y_b)\lambda_{acde}\lambda_{bcde}$$

- Contributions to beta-functions

$$\frac{\partial c}{\partial \lambda_{abcd}} \supset d_1 \text{tr}(y_a^* y_b y_c^* y_d) + 2d_2 \text{tr}(y_d^* y_e)\lambda_{abce} + \text{permutations}$$
$$\frac{\partial c}{\partial y_a} \supset 2d_1 y_b y_c^* y_d \lambda_{abcd} + d_2 y_b \lambda_{acde}\lambda_{bcde}$$

- True beta-functions

$$B_{abcd}^{\text{(one-loop)}} \supset -\frac{1}{16\pi^2}\text{tr}(y_a^* y_b y_c^* y_d) + \frac{1}{16\pi^2}\frac{1}{6}\text{tr}(y_d^* y_e)\lambda_{abce} + \text{permutations}$$
$$B_a^{\text{(two-loop)}} \supset -\frac{2}{(16\pi^2)^2} y_b y_c^* y_d \lambda_{abcd} + \frac{1}{(16\pi^2)^2}\frac{1}{12} y_b \lambda_{acde}\lambda_{bcde}$$

- $d_2/d_1 = -1/12$ for both beta-functions $\Rightarrow$ No obstruction to gradient flow interpretation at two loops !

Polchinski–Dorigoni–Rychkov argument at three loops

- Real scalar fields only   JFF, Grinstein, Stergiou (2012)
  - $\mathcal{Q}_{abcd}\,\mathrm{B}^{(\text{three-loop})}_{abcd} \neq 0$
  - Scale invariance does NOT imply conformal invariance
  - Obstruction due to several terms

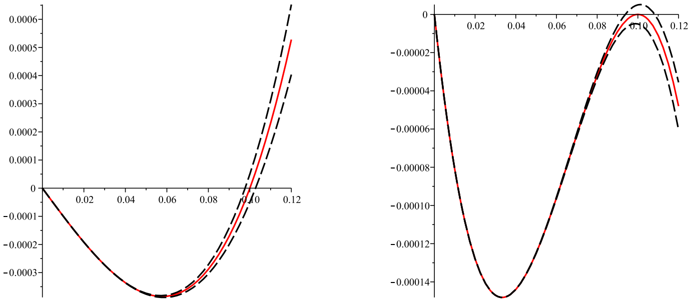
  $\Rightarrow$ Obstruction nevertheless allows for weakly-coupled gradient flow interpretation   Jack, Osborn (1990)

- Real scalar fields and Weyl fermions   *ibid*
  - $\mathcal{P}^{*}_{a|ij}\,\mathrm{B}^{(\text{three-loop})}_{a|ij} \neq 0$
  - Scale invariance does NOT imply conformal invariance
  - Obstruction due to several terms

  $\Rightarrow$?

Schematically,

- RG flow fixed point
  - Stable with respect to higher-order corrections
- RG flow recurrent behavior
  - Would-be RG flow fixed point at lowest order unstable with respect to higher-order corrections



Systematic approach

Scale-invariant trajectories at weak coupling

$$\lambda_{abcd} = \sum_{n\geq 1} \lambda_{abcd}^{(n)} \epsilon^n \qquad y_{a|ij} = \sum_{n\geq 1} y_{a|ij}^{(n-\frac{1}{2})} \epsilon^{n-\frac{1}{2}} \qquad g_A = \sum_{n\geq 1} g_A^{(n-\frac{1}{2})} \epsilon^{n-\frac{1}{2}}$$

$$Q_{ab} = \sum_{n\geq 3} Q_{ab}^{(n)} \epsilon^n \qquad P_{ij} = \sum_{n\geq 3} P_{ij}^{(n)} \epsilon^n$$

- ϵ small parameter
  - Obvious choice in $d = 4 - \epsilon$
  - One-loop gauge coupling beta-function coefficient in $d = 4$ Banks, Zaks (1982)
- Form of expansions determined by beta-functions
  - For coupling constants $\Rightarrow$ Lowest-order terms in beta-functions (would-be RG flow fixed points)
  - For virial current $\Rightarrow$ Higher-order terms in beta-functions due to Polchinski–Dorigoni–Rychkov argument and gradient flow interpretation

Examples ?

Unphysical $d = 4 - \epsilon$ case

- Two real scalars and two Weyl fermions
  - Limit cycle (bounded-from-below scalar potential, CP conservation, vacuum at origin of field space)
  - Strongly-coupled condensed matter example in $\epsilon \to 1$ limit (universality class ?)

$$
\begin{aligned}
V = {} & \tfrac{1}{24}\lambda_1\phi_1^4 + \tfrac{1}{24}\lambda_2\phi_2^4 + \tfrac{1}{4}\lambda_3\phi_1^2\phi_2^2 + \tfrac{1}{6}\lambda_4\phi_1^3\phi_2 + \tfrac{1}{6}\lambda_5\phi_1\phi_2^3 \\
& + [\tfrac{1}{2}y_1\phi_1(\psi_1\psi_1 - \psi_2\psi_2) + \tfrac{1}{2}y_2\phi_2(\psi_1\psi_1 - \psi_2\psi_2) + \text{h.c.}]
\end{aligned}
$$

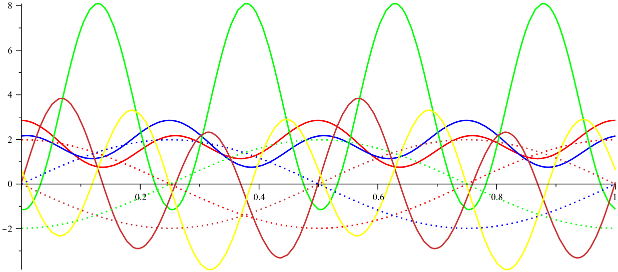
$$
Q = \begin{pmatrix} 0 & q_1 \\ -q_1 & 0 \end{pmatrix} \qquad P = \begin{pmatrix} ip_1 & p_3 + ip_4 \\ -p_3 + ip_4 & ip_2 \end{pmatrix}
$$

$d = 4 - \epsilon$ example with two real scalars and two Weyl fermions

$$\lambda_1 = \frac{8(7087+357\sqrt{52953})}{102885}\pi^2\epsilon \qquad \lambda_2 = \frac{64(6346+9\sqrt{52953})}{102885}\pi^2\epsilon \qquad \lambda_3 = -\frac{272(\sqrt{52953}-57)}{102885}\pi^2\epsilon$$

$$\lambda_4 = \frac{32\sqrt{323(757-3\sqrt{52953})}}{102885}\pi^2\epsilon \qquad \lambda_5 = \frac{272\sqrt{323(757-3\sqrt{52953})}}{102885}\pi^2\epsilon \qquad y_1 = \frac{2}{5}\sqrt{10}\pi\sqrt{\epsilon}$$

$$y_2 = 0 \qquad q_1 = \frac{\sqrt{323(757-3\sqrt{52953})}}{2057700}\epsilon^3 \qquad p_i = \{0, 0, 0, \text{undetermined}\}$$



$d = 4 - \epsilon$ example with two real scalars and two Weyl fermions

- Stability matrix eigenvalues x
  - Seven-dimensional coupling subspace $\Rightarrow$ Seven independent couplings $(\lambda_{1,\ldots,5}, y_{1,2})$
  - $\det(x\mathbb{1} - S) = 0$ with $x = z\epsilon + \cdots$

$$z(z-1)\left(z^5 - \frac{\sqrt{52953}}{57}z^4 + \frac{1894+\sqrt{52953}}{475}z^3 - \frac{240768-335\sqrt{52953}}{135375}z^2 - \frac{421203-1573\sqrt{52953}}{225625}z + \frac{136(757\sqrt{52953}-158859)}{64303125}\right) = 0$$

- Character of scale-invariant trajectory
  - IR attractive deformations
    $z \approx 2.4, \quad z = 1, \quad z \approx 0.99, \quad z \approx 0.74, \quad z \approx 0.095$
  - IR repulsive deformation $z \approx -0.19$
  - Deformation along scale-invariant trajectory $z = 0$

Physical $d = 4$ case

- $SU(3)$ gauge theory with two real scalars (singlet) and two active flavors of Weyl fermions (fundamental)
  - $(29 - 3\epsilon)/2$ sterile flavors of Weyl fermions with $\epsilon = 1/3$
  - Limit cycle (unbounded-from-below scalar potential, CP non-conservation)

$\Rightarrow$ Example with bounded-from-below scalar potential ?

What is happening ?

Quiz answers

Questions about scale invariance and conformal conformal invariance

✗ Do scale field theories (SFTs) live at RG flow fixed points ?

✗ Do conformal field theories (CFTs) live at RG flow fixed points ?

✓ Does scale invariance imply conformal invariance ?

- Proved to be true for $d = 2$ unitarity interacting quantum field theories (QFTs) with well-defined correlation functions Zamolodchikov (1986) & Polchinski (1988)
- Assumed to be true for $d = 4$

Question about possible types of RG flows

✓ Are there RG flow recurrent behaviors ?

c-theorem and gradient flow at weak coupling

- Weyl consistency conditions Osborn (1989,1991) & Jack, Osborn (1990)

$$\frac{\partial c(g)}{\partial g^I} = (G_{IJ} + A_{IJ})\beta^J \Rightarrow \frac{dc(g(t))}{dt} = -\beta^I G_{IJ}(g)\beta^J$$

  - Curved spacetime $\Rightarrow$ Background metric with spacetime-dependent couplings

$\Rightarrow$ (Weak-coupling) RG flow recurrent behaviors forbidden at all loops ?

Local and global renormalized operators

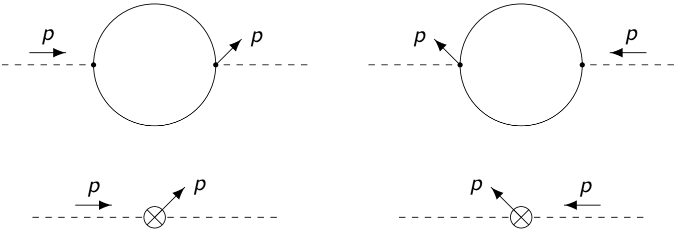
Global renormalized operator $\mathcal{O}_I(x) = \partial\mathcal{L}(x)/\partial g^I$

- Finite global insertion in Green functions $\Rightarrow$ $-i\partial\langle\ldots\rangle/\partial g^I = \langle\int d^dx\,\mathcal{O}_I(x)\ldots\rangle$
- Infinite local insertion in Green functions $\Rightarrow \langle\mathcal{O}_I(x)\ldots\rangle$

Local renormalized operator $[\mathcal{O}_I(x)] = \delta\mathcal{A}/\delta g^I(x)$

- Finite local insertion in Green functions $\Rightarrow$ $\langle[\mathcal{O}_I(x)]\ldots\rangle = \langle(\mathcal{O}_I(x) - \partial_\mu J_I^\mu(x))\ldots\rangle$
- Infinite current $J_I^\mu(x) = -(N_I)_{ab}\phi_a D^\mu\phi_b + (M_I)_{ij}\bar{\psi}_i i\bar{\sigma}^\mu\psi_j$
  - $(N_I)_{ba} = -(N_I)_{ab}$ and $(M_I)^*_{ji} = -(M_I)_{ij}$
  - $N_I = \sum_{i\geq 1}\frac{N_I^{(i)}}{\epsilon^i}$ and $M_I = \sum_{i\geq 1}\frac{M_I^{(i)}}{\epsilon^i}$

Computations of new divergences



$$(N_{c|ij})_{ab} = -\frac{1}{16\pi^2\epsilon}\,\frac{1}{2}(y^*_{a|ij}\delta_{bc} - y^*_{b|ij}\delta_{ac}) + \text{h.c.} + \text{finite}$$

Finite contributions to EM tensor

Anomalous trace Osborn (1989,1991) & Jack, Osborn (1990)

$$[\Theta_\mu{}^\mu(x)] = \beta^I[\mathcal{O}_I] - D_\mu[S_{ab}\phi_a D^\mu\phi_b - R_{ij}\bar{\psi}_i i\bar{\sigma}^\mu\psi_j] - ((\delta+\gamma)f)\cdot\frac{\delta}{\delta f}\mathcal{A}$$

$$f_0 = \mu^{(\frac{1}{2}-\delta)\epsilon}Z^{\frac{1}{2}}(g)f \qquad g_0^I = \mu^{k_I\epsilon}(g^I + L^I(g))$$

$$\hat{\gamma} = (\tfrac{1}{2}-\delta)\epsilon - k_I g^I\partial_I Z^{\frac{1}{2}(1)} \qquad \hat{\beta}^I = -k_I g^I\epsilon - k_I L^{I(1)} + k_J g^J\partial_J L^{I(1)}$$

$$S = -k_I g^I N_I^{(1)} \qquad R = -k_I g^I M_I^{(1)}$$

Ambiguities in RG functions

Relevant quantities Osborn (1989,1991) & Jack, Osborn (1990)

- Square root of wavefunction renormalization $Z^{\frac{1}{2}}$
  - Freedom $Z^{\frac{1}{2}} \to \tilde{Z}^{\frac{1}{2}} = OZ^{\frac{1}{2}}$ with $Z = Z^{\frac{1}{2}T} Z^{\frac{1}{2}} \to Z^{\frac{1}{2}T} O^T O Z^{\frac{1}{2}}$
  - $O^T O = 1$ and $O = 1 + \sum_{i \geq 1} \frac{O^{(i)}}{\epsilon^i}$
- Extra freedom with $\omega = k_I g^I \partial_I O^{(1)}$

$$Z^{\frac{1}{2}(1)} \to Z^{\frac{1}{2}(1)} + O^{(1)} \qquad L^{I(1)} \to L^{I(1)} - (gO^{(1)})^I \qquad N_I^{(1)} \to N_I^{(1)} - \partial_I O^{(1)}$$

$$\hat{\gamma} \to \hat{\gamma} - \omega \qquad \hat{\beta}^I \to \hat{\beta}^I - (g\omega)^I \qquad S \to S + \omega$$

- Invariant anomalous trace

$$[\Theta_\mu{}^\mu(x)] = (\beta^I + (gS)^I)[\mathcal{O}_I] - ((\delta + \gamma + S)f) \cdot \frac{\delta}{\delta f}\mathcal{A}$$

$$= \mathrm{B}^I[\mathcal{O}_I] - ((\delta + \Gamma)f) \cdot \frac{\delta}{\delta f}\mathcal{A}$$

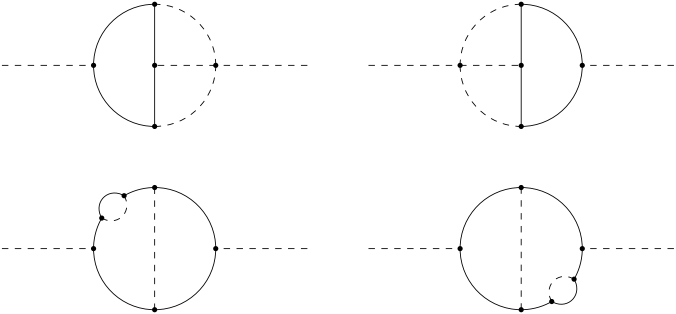
Necessary condition for conformality

- Generalized RG flow
  - $B^I = 0$
  - CFTs $\Rightarrow$ Generalized RG flow fixed points
- Usual RG flow
  - $\beta^I = -(gS)^I$
  - CFTs $\Rightarrow$ Usual RG flow recurrent behaviors and fixed points

$\Rightarrow$ Are examples of usual RG flow recurrent behaviors CFTs ?

Computation of S

$S^{(\text{one-loop})} = S^{(\text{two-loop})} = 0$ due to symmetry of contributions to N_I



$$(16\pi^2)^3 S_{ab} = \tfrac{5}{8}\mathrm{tr}(y_a\, y_c^*\, y_d\, y_e^*)\lambda_{bcde} + \tfrac{3}{8}\mathrm{tr}(y_a\, y_c^*\, y_d\, y_d^*\, y_b\, y_c^*) - \{a \leftrightarrow b\} + \text{h.c.}$$

$Q = S \Rightarrow$ Examples are CFTs ! JFF, Grinstein, Stergiou (2012)

Generalized c-theorem

- Weyl consistency conditions and local current conservation
  Osborn (1989,1991) & Jack, Osborn (1990)

$$\frac{\partial c(g)}{\partial g^I} = (G_{IJ} + A_{IJ})\,\mathrm{B}^J \Rightarrow \frac{dc(g(t))}{dt} = -\,\mathrm{B}^I G_{IJ}\,\mathrm{B}^J$$

  - Curved spacetime ⇒ Background metric with spacetime-dependent couplings
  - Spin-one operator of dimension 3 ⇒ Background gauge fields with gauge-dependent couplings

  ⇒ (Weak-coupling) RG flow recurrent behaviors allowed at all loops

- Scale invariance implies conformal invariance JFF, Grinstein, Stergiou (2012) & Luty, Polchinski, Rattazzi (2012)

Features and future work

Features of SFTs and CFTs

- Usual RG flow
  - Recurrent behaviors and fixed points
- Generalized RG flow
  - SFTs $\Rightarrow$ Recurrent behaviors
  - CFTs $\Rightarrow$ Fixed points
- Generalized c-theorem $\Rightarrow$ Only CFTs allowed
  - $\Rightarrow$ Scale invariance implies conformal invariance
  - Unexpected CFTs with expected behaviors

Future work

- Proof at strong coupling

Thank you !

Extra

Current conservation

- Divergence of current $J^\mu(x)$ without use of EOMs Collins (1984)

$$\partial_\mu J^\mu(x) = \Delta_{\text{EOM}} + \Delta_{\text{Classical}} + \Delta_{\text{Anomaly}}$$

- Green function of elementary fields with current $J^\mu(x)$ and Ward identity

  ✔ $\Delta_{\text{EOM}} \Rightarrow$ Expected contact terms from Ward identity
  ✗ $\Delta_{\text{Classical}} \Rightarrow$ Usual non-anomalous classical violation
  ✗ $\Delta_{\text{Anomaly}} \Rightarrow$ Possible anomalous violation in divergent Green functions

- Example: $\mathcal{L} = -\frac{1}{4}F^A_{\mu\nu}F^{A\mu\nu} + \bar{\psi}_i(i\gamma^\mu D_\mu \delta_{ij} - M_{ij})\psi_j$
  - Vector current $J_V^{\mu a}(x) = \bar{\psi}\gamma^\mu t^a \psi$ with $\Delta_{\text{EOM}} \neq 0$, $\Delta_{\text{Classical}} = i\bar{\psi}[M, t^a]\psi$ and $\Delta_{\text{Anomaly}} = 0$
  - Axial current $J_A^{\mu a}(x) = \bar{\psi}\frac{1}{2}[\gamma^\mu, \gamma^5]t^a\psi$ with $\Delta_{\text{EOM}} \neq 0$, $\Delta_{\text{Classical}} = i\bar{\psi}\gamma^5\{M, t^a\}\psi$ and $\Delta_{\text{Anomaly}} = \frac{1}{2}\bar{\psi}\{\gamma^\mu, \gamma^5\}t^a D_\mu\psi - \frac{1}{2}D_\mu\bar{\psi}\{\gamma^\mu, \gamma^5\}t^a\psi$