DISENTANGLING THE OVERLAPPING SINGULARITIES OF NNLO AMPLITUDES

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Mainz Theorie Palaver, 26.10.2010

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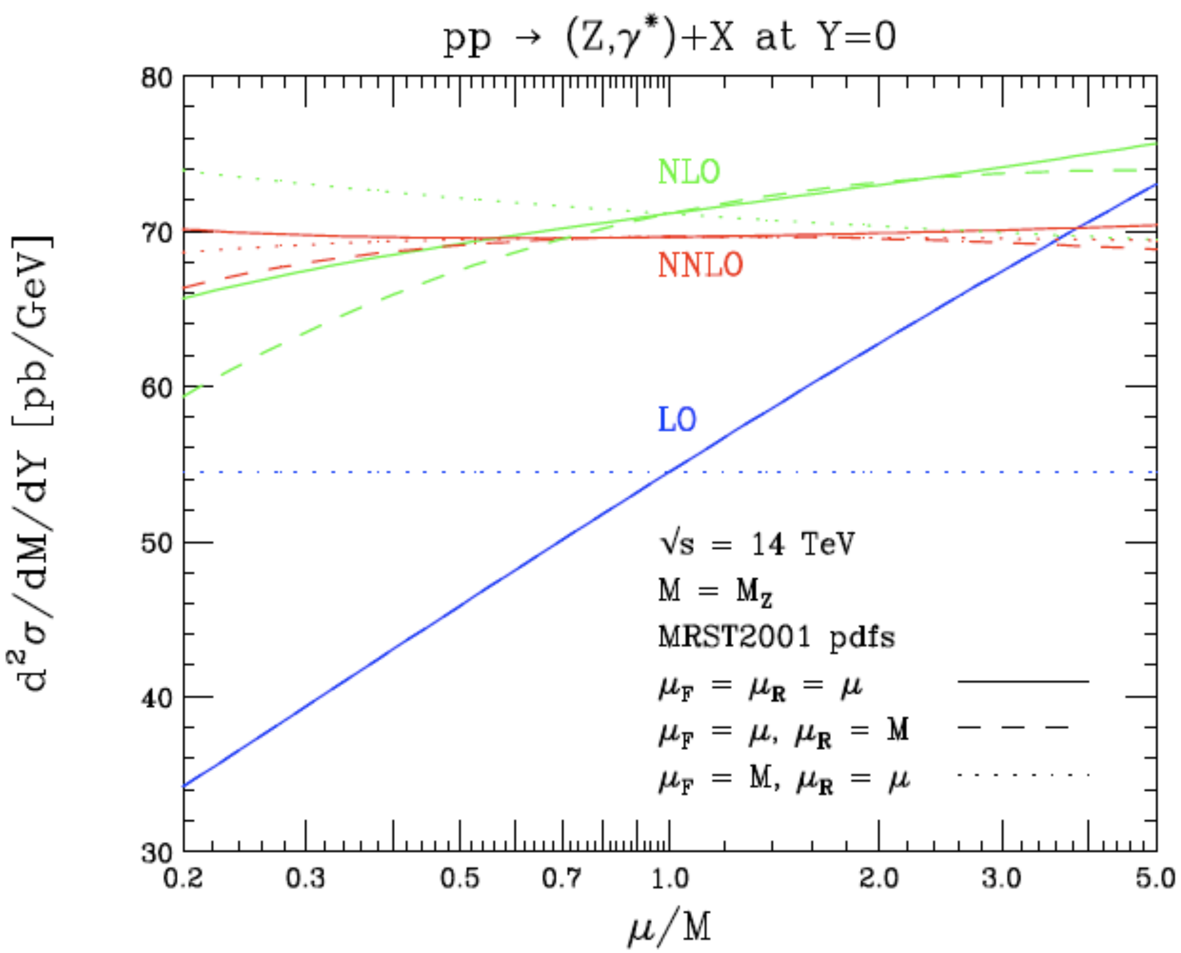
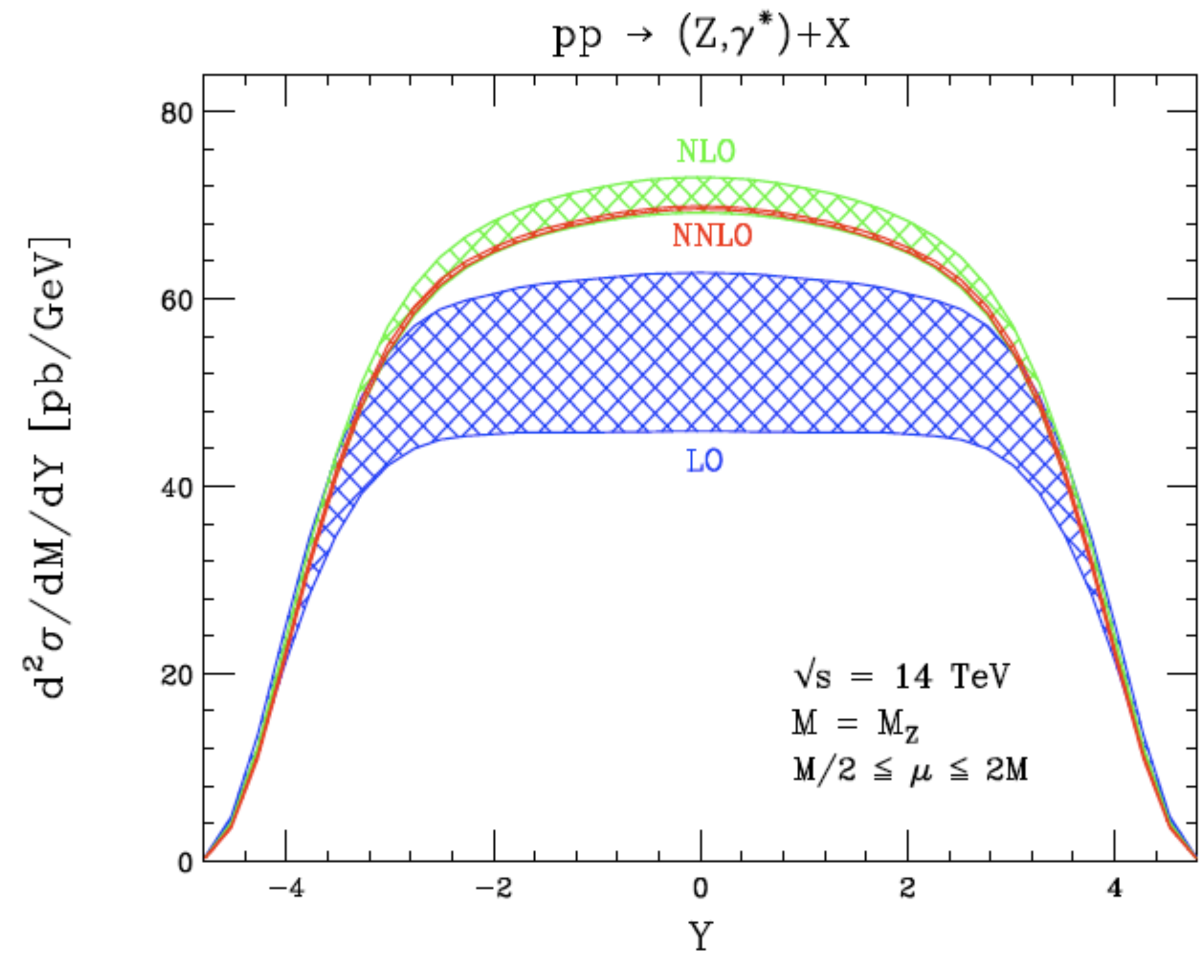
Motivation: why bother with NNLO fixed order calculations ?

Why NNLO: to stabilize cross sections with respect to scale variation





Z boson production: central rapidity bin with varying scales and rapidity distribution for the LHC

[Anastasiou, Dixon, Melnikov, Petriello, hep-ph/0312266]

Why NNLO: to stabilize cross sections with respect to scale variation

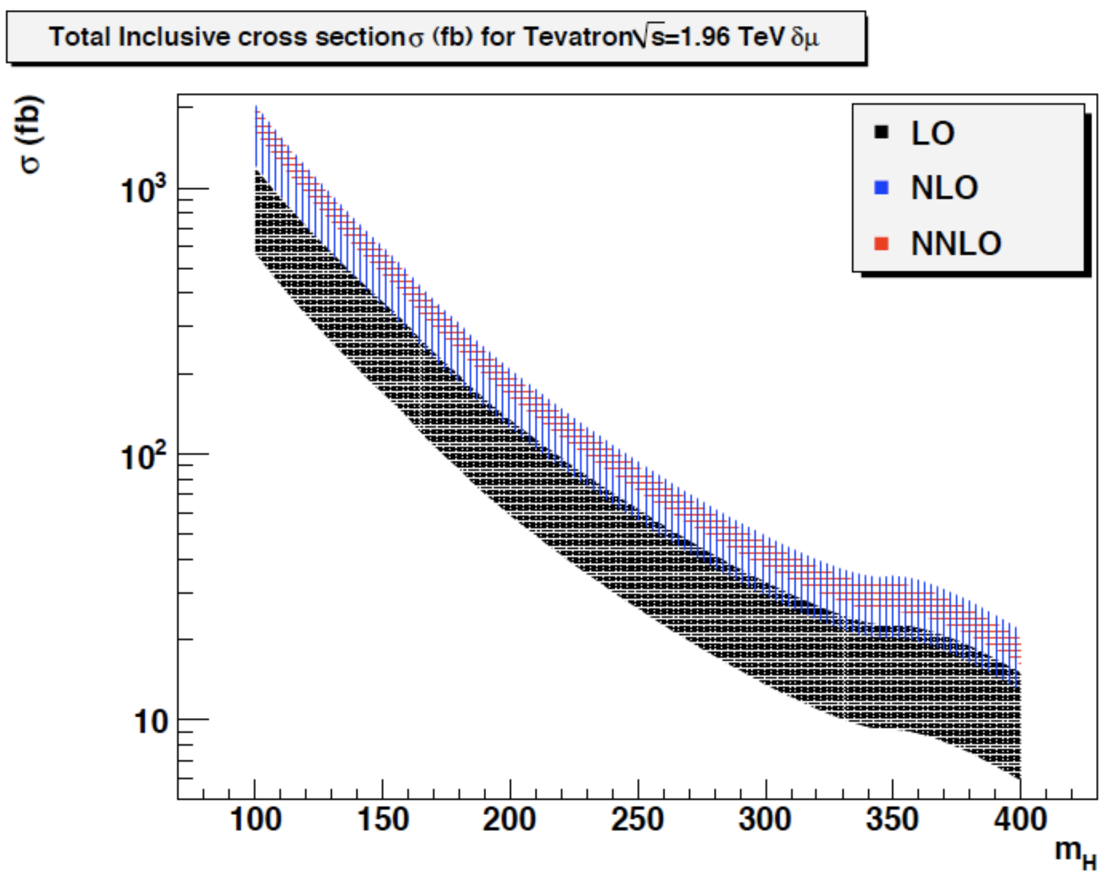


Figure 3: Total inclusive cross section for Tevatron ($\sqrt{s}$ = 1.96 TeV). The bands represen total uncertainty due to scale variation ($\mu_F = \mu_R \in [m_H/4, m_H]$).

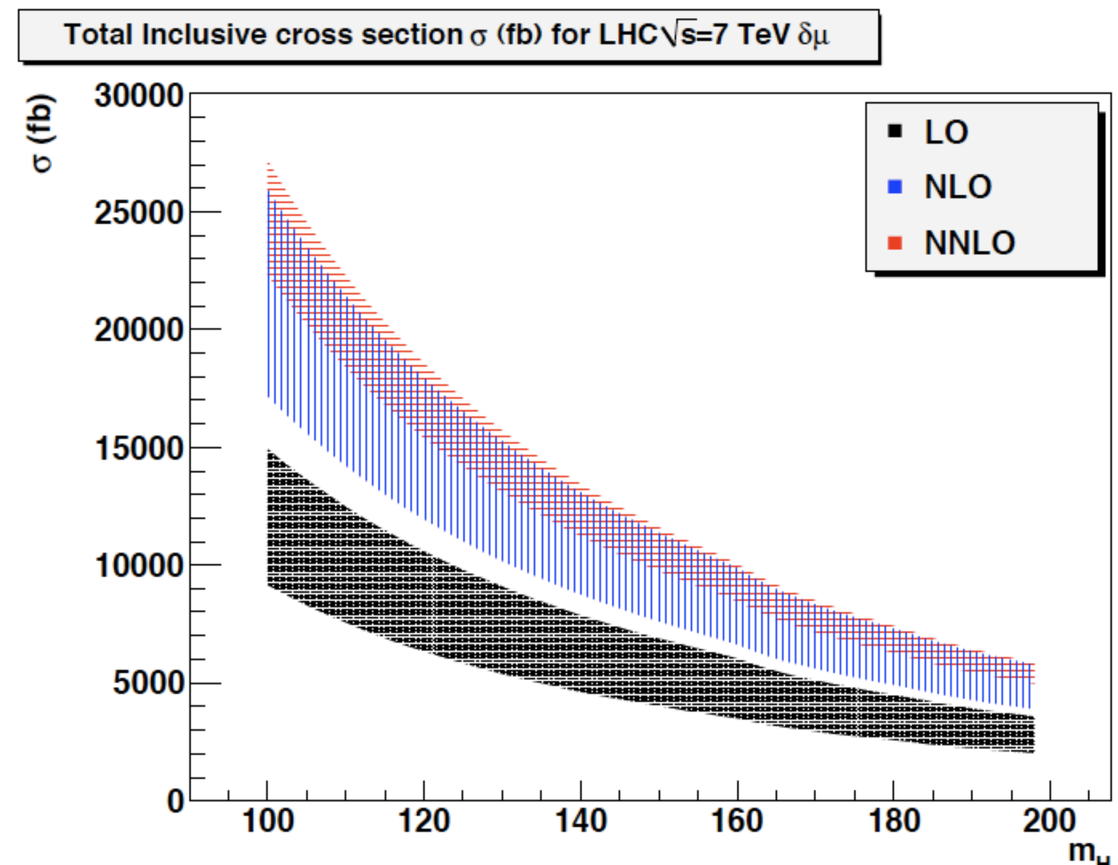


Figure 7: Total inclusive cross section for LHC ($\sqrt{s}$ = 7 TeV). The bands represent the total uncertainty due to scale variation ($\mu_F = \mu_R \in [m_H/4, m_H]$).

Plots produced with **fehipro**

Higgs @NNLO: Anastasiou, Melnikov, Petriello **hep-ph/0409088, hep-ph/0501130.** Catani, Grazzini**, hep-ph/0703012, 0801.3232, ...**

Why NNLO: smaller theoretical uncertainty for measured quantities.

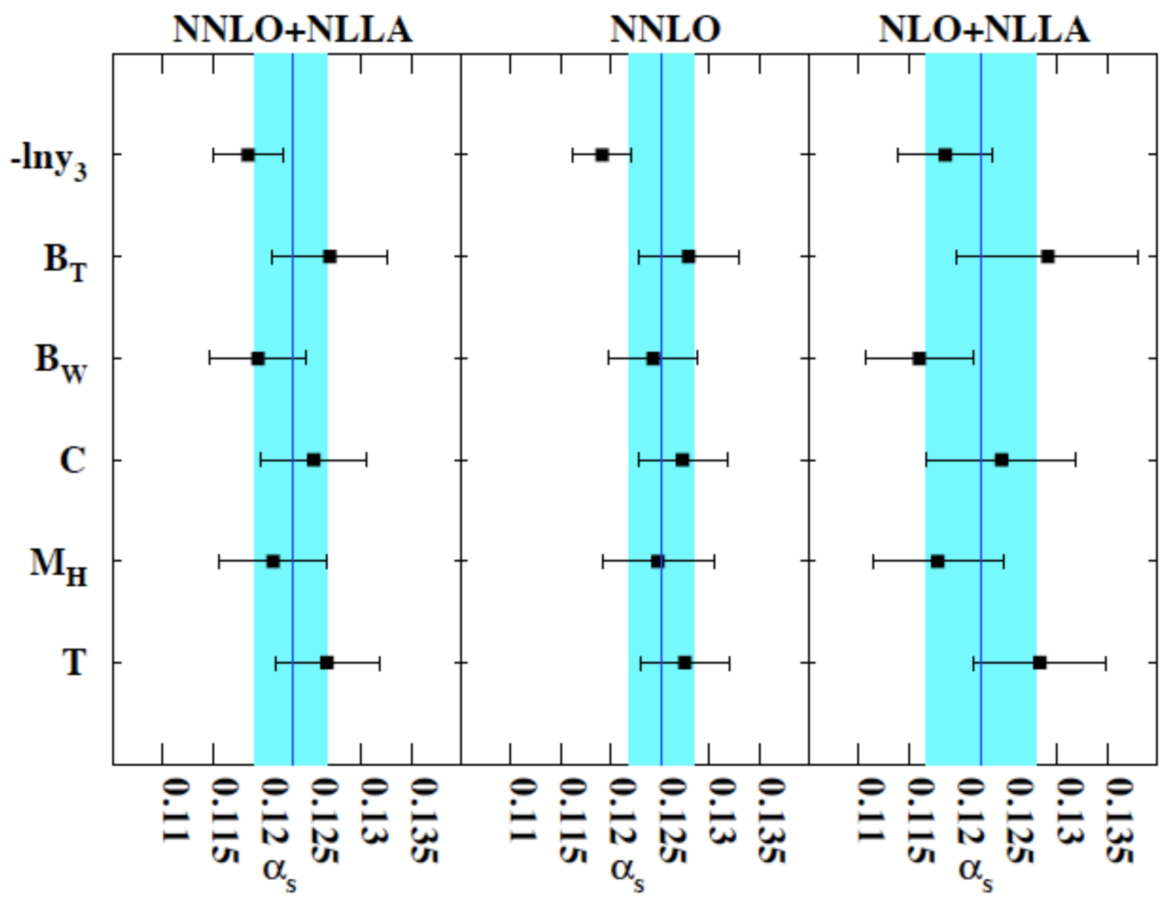


Figure 6: The measurements of the strong coupling constant α_s for the six event shapes, at $\sqrt{s} = M_Z$, when using QCD predictions at different approximations in perturbation theory. The shaded area corresponds to the total uncertainty, as in Fig. 5.

$$\alpha_s(M_Z) = 0.1175 \pm 0.0020\,(\mathrm{exp}) \pm 0.0015\,(\mathrm{theo})$$

(from 3 jet rate)

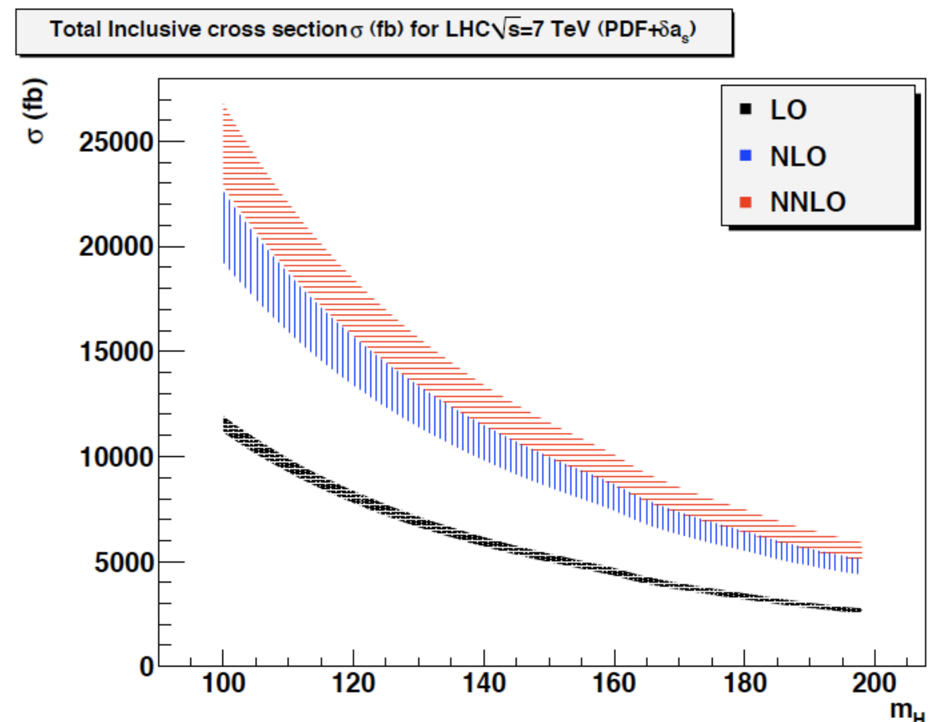
$$a_s \text{ from } e^+e^- \to 3j$$

Dissertori, Gehrmann-deRidder, Gehrmann, Glover, Heinrich, Luisoni, Stenzel 0910.4283, 0906.3436

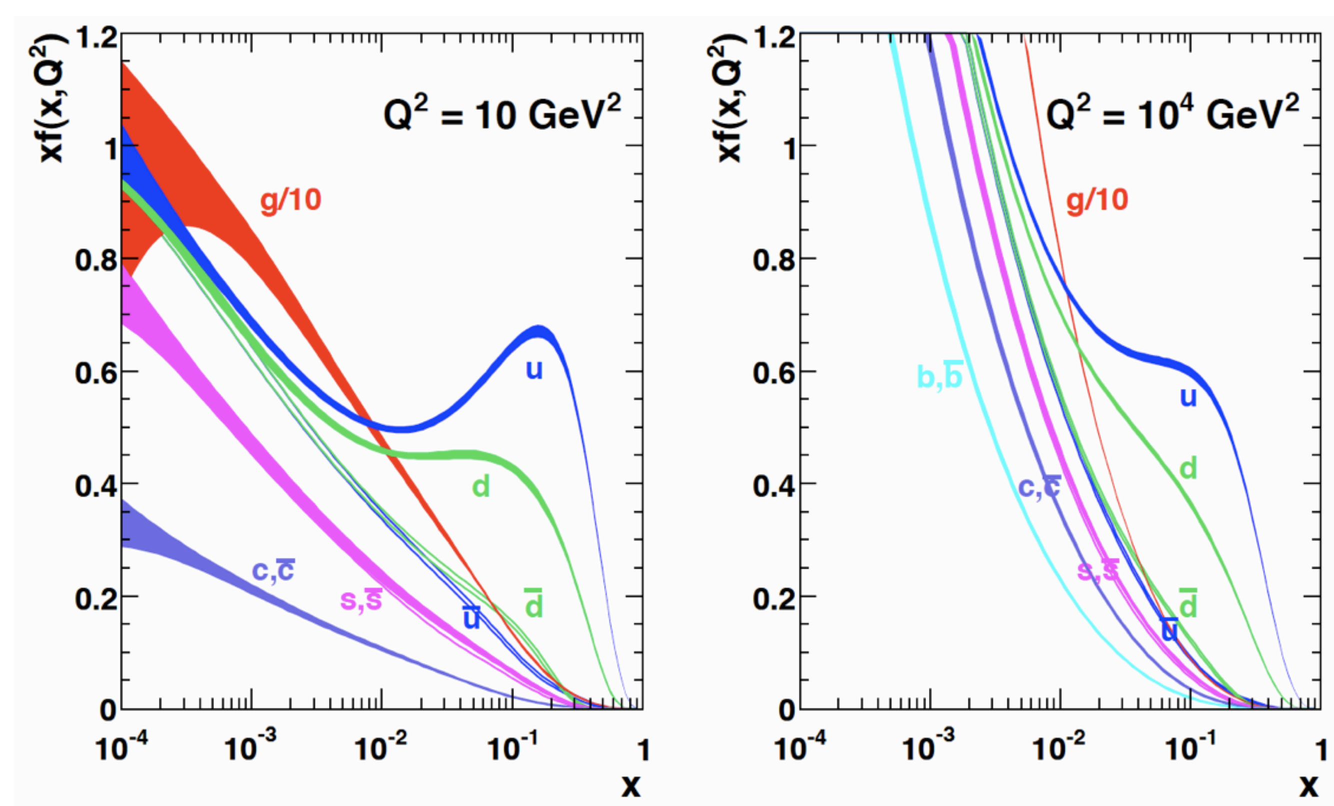
$e^+e^- \to 3j$ Gehrmann-de Ridder, Gehrmann, Glover, Heinrich 0711.4711, Weinzierl 0807.3241

Why NNLO: To constrain the gluon PDF

- Gluon PDF uncertainty is critical
- MSTW2008 include Run II jet data in their NNLO PDF fit assuming that the unknown NNLO corrections are small
- Probably legitimate, but other groups prefer to wait for the the full NNLO calculation.
- Further jet data would constrain the ubiquitous gluon pdf



MSTW 2008 NNLO PDFs (68% C.L.)



NNLO necessary ingredients

- NNLO splitting functions (Moch, Vermaseren, Vogt)
- NNLO PDFs (MSTW2008, ABKM2009, well restrained at interesting region)
- Matrix elements for double virtual, virtual-real, virtual squared, double real
- Suitable parameterization for double real
- Fully differential cross-sections
- Reasonably fast final product code.

NNLO strategies for double virtual

- Reduction to master integrals by ibp equations (note that a generic basis of masters is not known at two loops).
- Calculate the master integrals analytically or numerically by
  - ★ Mellin-Barnes
  - ★ Differential equations
  - ★ Sector decomposition

NNLO strategies for double real

- Antenna subtraction
- qT subtraction
- NNLO subtraction along the lines of NLO
- NNLO subtraction + sector decomposition
- Sector decomposition

The “bottleneck”

In both double virtual and double real one has to deal with integrals with a complicated singularity structure that has to be factorized

Singularity structure: the simplest possible case

$$\int \frac{dx_1 \dots dx_N}{x_1^{1-\epsilon}} \frac{N(x1, \dots, x_n)}{D(x_1, \dots, x_n)}$$

Easy: we can subtract the singularity and Taylor expand over epsilon

$$I = \int_0^1 d^N\vec{x} \frac{f(x_1, x_2, \dots, x_N) - f(0, x_2, \dots, x_N)}{x_1} \left[\sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \log^n(x_1) \right]$$

$$+ \frac{1}{\epsilon} \int_0^1 d^N\vec{x} f(0, x_2, \dots, x_N).$$

Singularity structure: the next-to-simplest case

$$\int \frac{dx_1 \ldots dx_N}{x_1^{1-\epsilon} \ldots x_k^{1-\epsilon}} \frac{N(x1, \ldots, x_n)}{D(x_1, \ldots, x_n)}$$

We can define the + distribution expansion

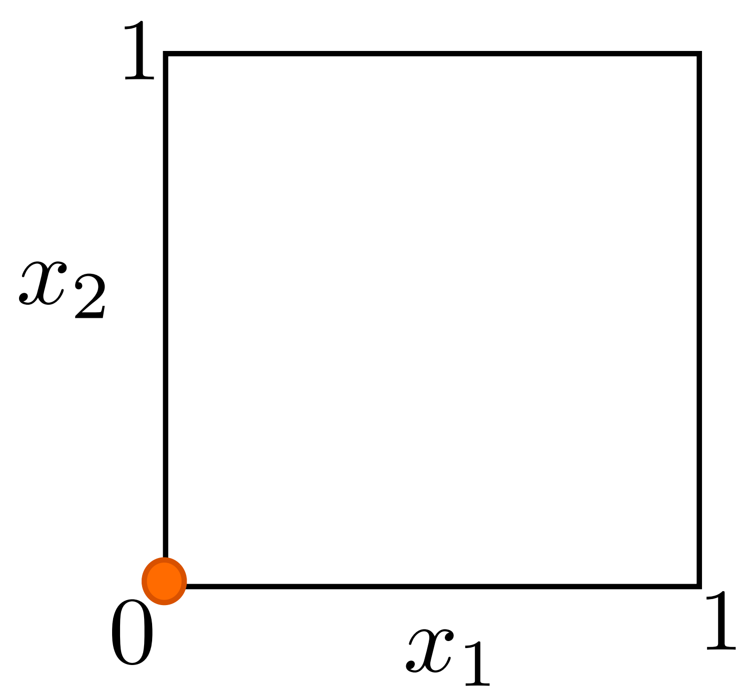
$$x_i^{-1+a_i\epsilon} = \frac{\delta(x_i)}{a_i\epsilon} + \sum_{n=0}^{\infty} \frac{a_i^n \epsilon^n}{n!} \left[\frac{\log^n(x_i)}{x_i}\right]_+$$

and expand the integrand for each variable

Singularity structure: overlapping singularity

$$\int dx_1 dx_2 \frac{1}{(x_1 + c x_2)^{2-\epsilon}}$$

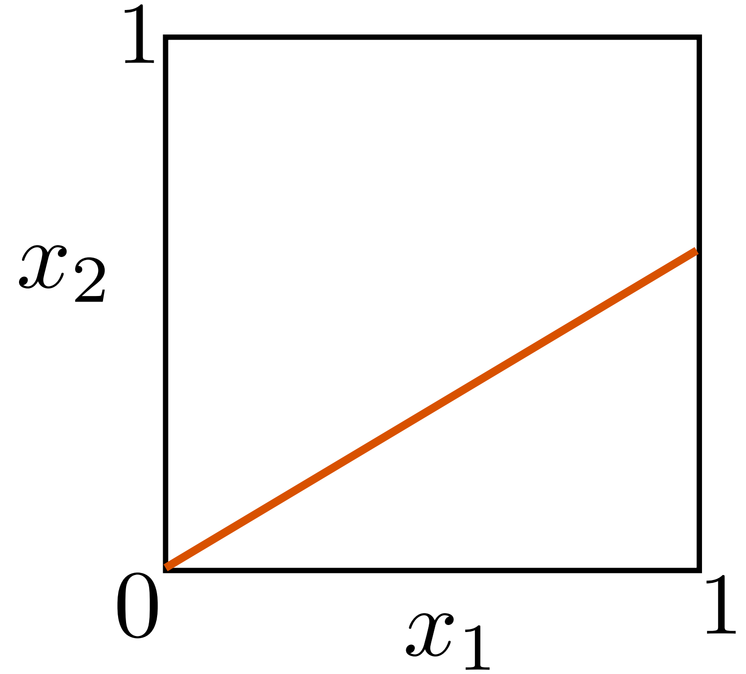
We have here an overlapping singularity at $x_1 = 0 \quad x_2 = 0$



Singularity structure: line singularity

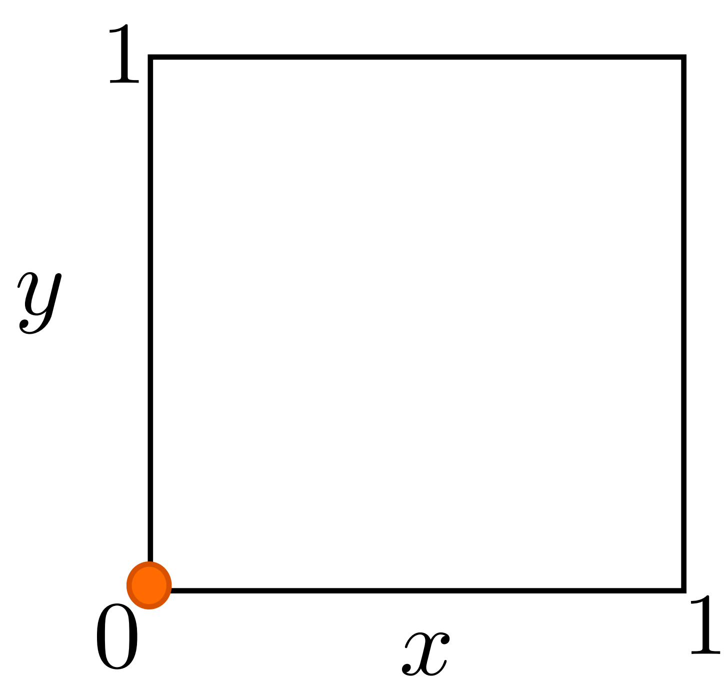
$$\int dx_1 dx_2 \frac{1}{(x_1 - cx_2)^{2-\epsilon}}$$

The integral is singular over a line in the integration region

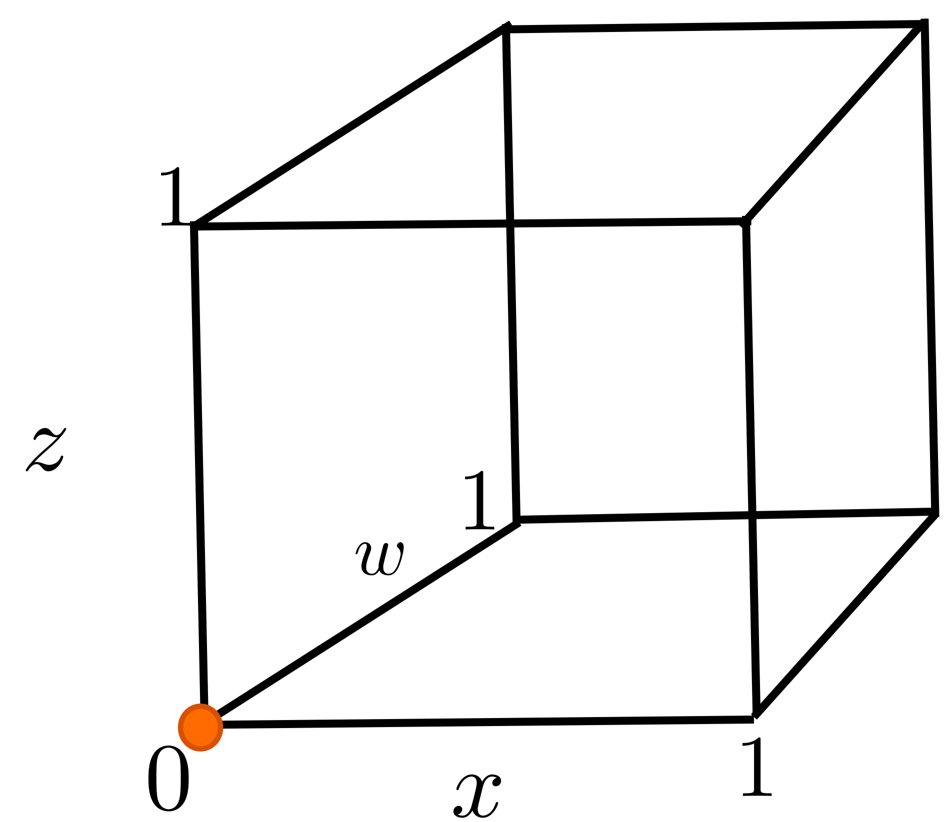


Singularity structure: entangled overlapping singularity

$$\int_0^1 dx\,dy\,dz\,dw \frac{(xyzw)^\epsilon}{(x+y(z+w))^2}$$



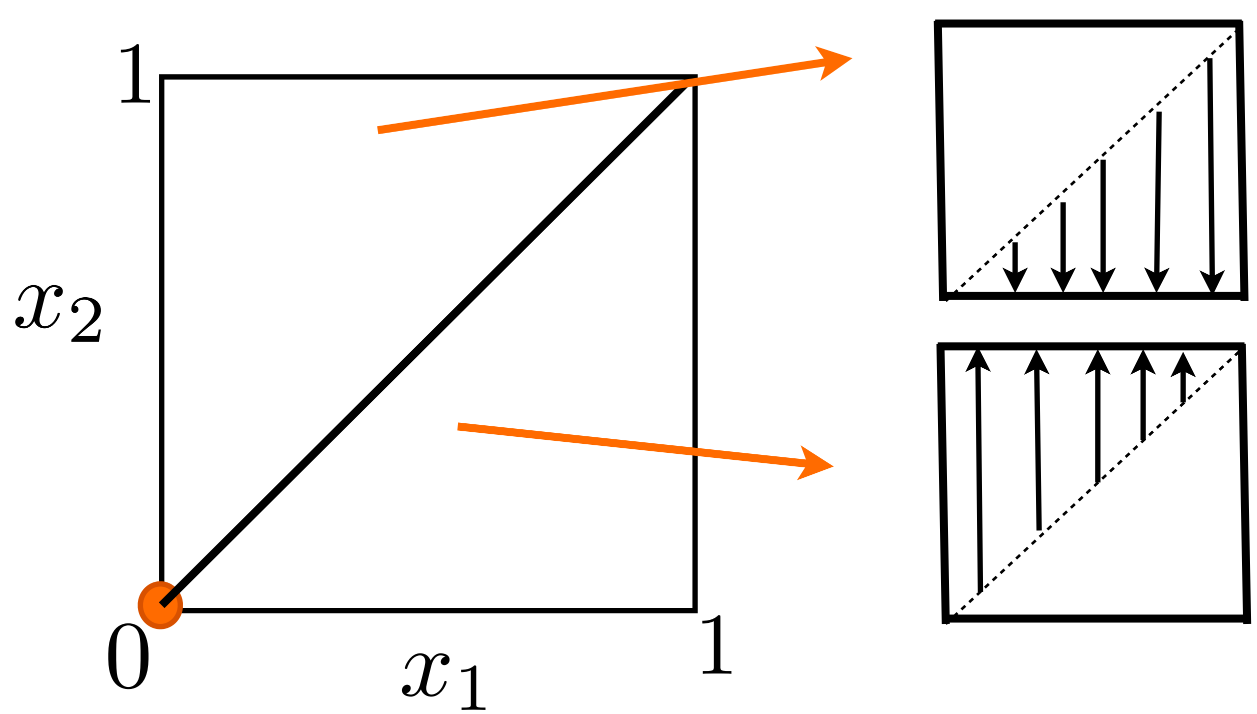
or



Factorizing singularities with sector decomposition

Sector decomposition employed by Hepp, but modern iterative version introduced by Binoth and Heinrich **hep-ph/0305234**

$$\int dx_1 dx_2 \frac{1}{(x_1 + c x_2)^{2-\epsilon}}$$



$$x_2 > x_1 \quad x_1 = x_2' x_1'$$

$$\int dx_1 dx_2 \frac{1}{x_2^{1-\epsilon}(x_1 + c)^{2-\epsilon}}$$

$$x_1 > x_2 \quad x_2 = x_2' x_1'$$

$$\int dx_1 dx_2 \frac{1}{x_1^{1-\epsilon}(1 + c x_2)^{2-\epsilon}}$$

This can be iterated ad nauseam, until all singularities are factorized

Factorizing singularities with sector decomposition

+	-
• Well established algorithm • Works for many loops • Works even better for real emission	• Deeply nested decompositions lead to proliferation of sectors • Numerical cancelations induced as the number of sectors grows

Threshold singularities cannot be removed by sector decomposition, but can be avoided numerically by contour deformation
A.L., Melnikov, Petriello hep-ph/0703273 , Anastasiou, Beerli, Daleo hep-ph/0703282

NEW: factorizing singularities with non-linear mappings

Factorizing singularities with non-linear mappings

$$I = \int_0^1 dx_1 dx_2 \frac{J(x_1, x_2)}{(x_1 + x_2)^{2+\epsilon}}$$

$$x_2 = \frac{x_1 x_2'}{x_1 + (1 - x_2')} \qquad \frac{\partial x_2}{\partial x_2'} = \frac{x_1(1 + x_1)}{[x_1 + (1 - x_2')]^2}$$

$$I = \int_0^1 dx_1 dx_2' \frac{(1 - x_2' + x_1)^\epsilon J(x_1, \frac{x_1 x_2'}{x_1 + 1 - x_2'})}{x_1^{1+\epsilon}(1 + x_1)^{1+\epsilon}}$$

The one loop massless, on-shell box as a toy example

$$I = \int \frac{d^d k}{i\pi^{\frac{d}{2}}} \frac{1}{k^2 (k+p_1)^2 (k+p_1+p_2)^2 (k+p_1+p_2+p_3)^2}$$

$$I = \int_0^1 dx_1 \ldots dx_4 \delta\left(1 - x_1 - \ldots - x_4\right) f(x_1, \ldots, x_4)$$

$$f(x_1, \ldots, x_4) \equiv \frac{\Gamma(2+\epsilon)}{\left[-s x_1 x_3 - t x_2 x_4 - i0\right]^{2+\epsilon}}.$$

The one loop massless, on-shell box as a toy example

$$I = 2\,\Gamma(2+\epsilon)\int_0^1 dy_1 dy_2 dy_3\,(1+y_1+y_2+y_3)^{2\epsilon}$$
$$\times\left\{[-sy_1 - ty_2y_3]^{-2-\epsilon} + [-ty_1 - sy_2y_3]^{-2-\epsilon}\right\}$$

Sector decomposition

$$\Theta(y_1 < y_2y_3):\qquad y_1 \to y_1y_2y_3$$
$$\Theta(y_2y_3 < y_1 < y_2):\qquad y_1 \to y_1y_2 \text{ and } y_3 \to y_3y_1$$
$$\Theta(y_2 < y_1):\qquad y_2 \to y_2y_1$$

The one loop massless, on-shell box as a toy example

$$I = I_1 + I_2 + I_3$$

$$I_1 = 2\,\Gamma(2+\epsilon)\int_0^1 dy_1 dy_2 dy_3\,(1 + y_1 y_2 y_3 + y_2 + y_3)^{2\epsilon} \times \left\{[-sy_1 - t]^{-2-\epsilon} + [-ty_1 - s]^{-2-\epsilon}\right\}(y_2 y_3)^{-1-\epsilon}$$

$$I_2 = 2\,\Gamma(2+\epsilon)\int_0^1 dy_1 dy_2 dy_3\,(1 + y_1 y_2 + y_2 + y_3 y_1)^{2\epsilon} \times \left\{[-s - ty_3]^{-2-\epsilon} + [-t - sy_3]^{-2-\epsilon}\right\}(y_2 y_1)^{-1-\epsilon}$$

$$I_3 = 2\,\Gamma(2+\epsilon)\int_0^1 dy_1 dy_2 dy_3\,(1 + y_1 + y_2 y_1 + y_3)^{2\epsilon} \times \left\{[-s - ty_2 y_3]^{-2-\epsilon} + [-t - sy_2 y_3]^{-2-\epsilon}\right\} y_1^{-1-\epsilon}$$

The one loop massless, on-shell box as a toy example

$$I = 2\,\Gamma(2+\epsilon)\int_0^1 dy_1 dy_2 dy_3\,(1+y_1+y_2+y_3)^{2\epsilon}$$
$$\times\left\{[-sy_1 - ty_2y_3]^{-2-\epsilon} + [-ty_1 - sy_2y_3]^{-2-\epsilon}\right\}$$

$$y_1 \to \frac{y_1y_2y_3}{1-y_1+y_2y_3}$$

$$I = 2\,\Gamma(2+\epsilon)\int_0^1 dy_1dy_2dy_3(y_2y_3)^{-1-\epsilon}(1-y_1+y_2y_3)^{-\epsilon}\left[y_1y_2y_3+(1+y_2+y_3)(1-y_1+y_2y_3)\right]^{2\epsilon}$$
$$\times\left\{[-sy_1 - t(1-y_1) - ty_2y_3]^{-2-\epsilon} + [-ty_1 - s(1-y_1) - sy_2y_3]^{-2-\epsilon}\right\} \qquad (3.7)$$

Other typical examples

$$x \mapsto A \quad \equiv \quad x \to \frac{xA}{1-x+A}$$

$$I_2 = \int_0^1 dx\, dy\, dz\, \frac{(xyz)^\epsilon}{(x+yz)^N} \qquad x \mapsto yz \qquad I_2 = \int_0^1 \frac{dx\, dy\, dz}{(yz)^{N-1-2\epsilon}} \frac{(1-x+yz)^{N-2-\epsilon} x^\epsilon}{(1+yz)^{N-1}}$$

$$I_3 = \int_0^1 dx\, dy\, dz \frac{(xyz)^\epsilon}{(x+y+z)^3} \qquad y \mapsto x \;\; , \;\; z \mapsto x \quad I_3 = \int_0^1 \frac{dx\, dy\, dz}{x^{1-3\epsilon}} \frac{(1+x)^2(1-y+x)(1-z+x)(yz)^\epsilon}{((1+x)^2 - zy)^3}$$

$$I_4 = \int_0^1 dx\, dy\, dz\, dw \frac{(xyzw)^\epsilon}{(x+y(z+w))^2} \qquad x \mapsto y \;\; , \;\; z \mapsto x \;\; , \;\; w \mapsto x$$

$$I_4 = \int_0^1 \frac{dx\, dy\, dz\, dw}{y^{2-2\epsilon} x^{1-3\epsilon}} \frac{(zw)^\epsilon (1+y)(1+x)^2 (1-w+x)^{1-\epsilon} (1-z+x)^{1-\epsilon} (1-x+y)^{-\epsilon}}{\left[(x+1)^2 + (x+1)(x-y)(z+w) + (-1+2y-2x)zw\right]^3}$$

$$I_5 = \int_0^1 dx\, dy\, dz\, dw \frac{(xyzw)^\epsilon}{(x+y+zw)^3} \qquad x \mapsto zw \;\; , \;\; y \mapsto zw$$

$$I_5 = \int_0^1 dx\, dy\, dz\, dw \frac{(xy)^\epsilon}{z^{1-3\epsilon} w^{1-3\epsilon}} \frac{(1+zw)^2 (1-x+zw)^{1-\epsilon} (1-y+zw)^{1-\epsilon}}{(1-xy+2zw+z^2w^2)^3}$$

Active and passive singularities

$$I = \int_0^1 dxdy \frac{(xy)^\epsilon}{x(x+y)}$$

ACTIVE PASSIVE

$$x \to \frac{xy}{1-x+y}$$

$$y \to \frac{yx}{1-y+x}$$

$$I = \int_0^1 dxdy x^{-1+\epsilon} y^{-1+2\epsilon} (1-x+y)^{-\epsilon}$$

$$I = \int_0^1 dxdy \frac{y^{-\epsilon}}{x^{1+\epsilon}(1-y+x)^{1+\epsilon}}$$

MAP THE ACTIVE: OK

MAP THE PASSIVE: **NOT** OK

More complicated example with active/passive singularities

$$I = \int_0^1 dxdydz \frac{(xyz)^\epsilon}{xy(xy+z)}$$

$$x \to \frac{xz}{1-x+zx}$$

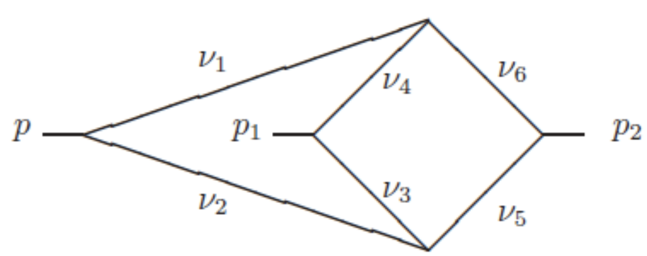
$$I = \int_0^1 dxdydz \frac{(xy^2z)^\epsilon(1-x+zx)^{-\epsilon}}{xyz(x(y+z)+(1-x))}$$

$$y \to \frac{y(1-x)}{1-y+(1-x)y} \qquad z \to \frac{z(1-x)}{1-z+(1-x)z}$$

$$I = \int_0^1 dxdydz \frac{(xy)^{-1+\epsilon}((1-x)z)^{-1+2\epsilon}}{(1-xy)^\epsilon(1-xz)^\epsilon(1-x^2yz)}$$

Applications $2 \longrightarrow 1$ in processes

Applications in double virtual: the non-planar two-loop triangle



$1 : k_1$

$2 : k_1 + p_1$

$3 : k_2$

$4 : k_2 + p_2$

$5 : k_1 + k_2$

$6 : k_1 + k_2 + p_1 + p_2$

$$Xtri = 4^{2+2\epsilon} \int_0^1 dx_1 dx_2 dz dy dx \frac{z y^{1+\epsilon}(1-y)^{-1-\epsilon}(1-z)^{-1-\epsilon}}{[x(1-x) + yz(x-x_1)(x-x_2)]^{2+2\epsilon}}$$

singularities at $x = 0$ or $x = 1$

$$x \to x/2 \qquad x \to 1 - x/2$$

$$Xtri = 4^{2+2\epsilon} \int_0^1 dx_1 dx_2 dz dy dx \frac{z y^{1+\epsilon}(1-y)^{-1-\epsilon}(1-z)^{-1-\epsilon}}{[x(2-x) + yz(2x_1-x)(2x_2-x)]^{2+2\epsilon}}$$

$$y \to y/2 \qquad z \to z/2 \qquad Xtri_{11} = 2^{6+9\epsilon} \int_0^1 dx_1 dx_2 dz dy dx \frac{z y^{1+\epsilon}(2-y)^{-1-\epsilon}(2-z)^{-1-\epsilon}}{[4x(2-x) + yz(2x_1-x)(2x_2-x)]^{2+2\epsilon}}$$

$$y \to y/2 \qquad z \to 1 - z/2 \qquad Xtri_{12} = 2^{6+9\epsilon} \int_0^1 dx_1 dx_2 dz dy dx \frac{(2-z) y^{1+\epsilon}(2-y)^{-1-\epsilon} z^{-1-\epsilon}}{[4x(2-x) + y(2-z)(2x_1-x)(2x_2-x)]^{2+2\epsilon}}$$

$$y \to 1 - y/2 \qquad z \to z/2 \qquad Xtri_{21} = 2^{6+9\epsilon} \int_0^1 dx_1 dx_2 dz dy dx \frac{z (2-y)^{1+\epsilon} y^{-1-\epsilon}(2-z)^{-1-\epsilon}}{[4x(2-x) + (2-y)z(2x_1-x)(2x_2-x)]^{2+2\epsilon}}$$

$$y \to 1 - y/2 \qquad z \to 1 - z/2 \qquad Xtri_{22} = 2^{6+9\epsilon} \int_0^1 dx_1 dx_2 dz dy dx \frac{(2-z)(2-y)^{1+\epsilon} y^{-1-\epsilon} z^{-1-\epsilon}}{[4x(2-x) + (2-y)(2-z)(2x_1-x)(2x_2-x)]^{2+2\epsilon}}$$

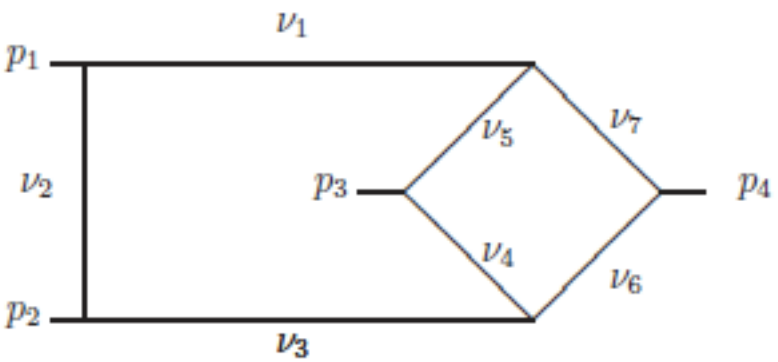
Applications in double virtual: the non-planar two-loop triangle

$Xtri$

- $Xtri_{11}$ $\qquad x \mapsto yzx_1x_2$
- $Xtri_{12}$ $\qquad x \mapsto yx_1x_2$
- $Xtri_{21}$ $\qquad x \mapsto zx_1x_2$
- $Xtri_{22}$ $\qquad x + x_1x_2 - x(x_1+x_2)$
  - $Xtri_{2211}$ $\qquad x \mapsto x_1x_2$
  - $Xtri_{2212}$ $\qquad x_1 \mapsto x \;,\; y \mapsto x_2 \;,\; z \mapsto x_1 \;,\; x_1 \mapsto x_2$
  - $Xtri_{2221}$ $\qquad x_2 \mapsto x \;,\; y \mapsto x_1 \;,\; z \mapsto x_2 \;,\; x_2 \mapsto x_1$
  - $Xtri_{2222}$ $\qquad$ finite

7 integrals: an improvement by a factor of 14 compared to sector decomposition.

Applications in double virtual: The non-planar double box



$$Xbox = C_\epsilon \int \frac{dx_1 dx_2 dx_3 dx_4 \delta(1 - x_1 - x_2 - x_3 - x_4) d\tau_1 d\tau_2 \; x_2^{1+\epsilon}}{(x_1 x_3 s + x_2 x_4 t_c + x_1 x_2 Q^2 + x_2 x_3 Q_t^2 + 0 x_1 x_4 (s + t + u))^{3+2\epsilon}}$$

where

$$Q_t^2 = (1 - \tau_1)(1 - \tau_2)s, \quad Q^2 = \tau_1 \tau_2 s, \quad t_c = \tau_2 (1 - \tau_1) u + (1 - \tau_2) \tau_1 t$$

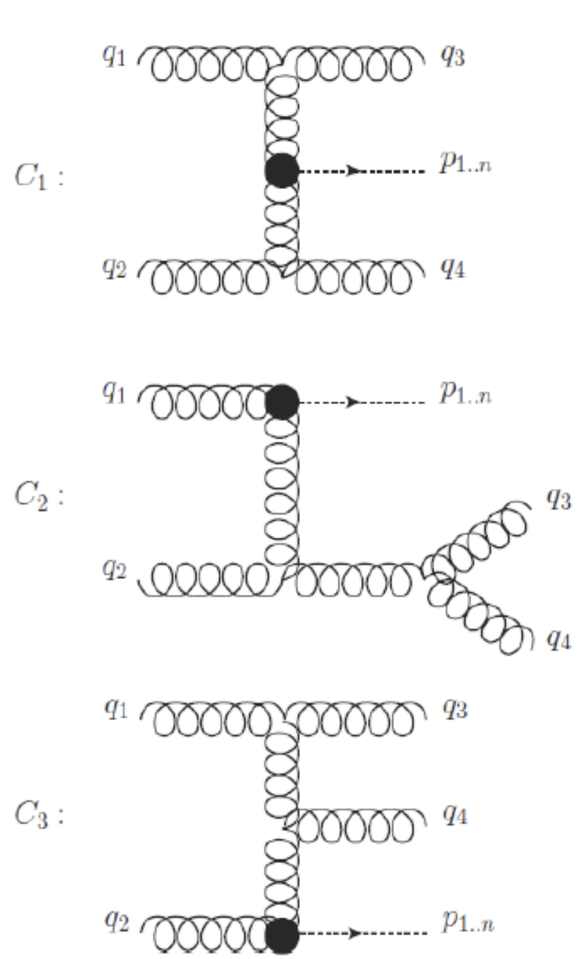
Note: this is a new representation inspired by Wilson loop considerations (see paper by Anastasiou and Banfi to be published soon)

More complicated structure resulting at 19 integrals

This is an improvement with a factor of 5 (at least) compared to sector decomposition

Applications in double real: Higgs revisited

Higgs + 2 gluons topologies



C_1: $\sim 1/(s_{13}s_{24})$

C_2: $\sim 1/(s_{34}s_{234})$

C_3: $\sim 1/(s_{134}s_{13})$

1. C_1^2:

$$\int \frac{d\Phi_3 N(\{s_{ij}\})}{(s_{13}s_{24})^2}, \int \frac{d\Phi_3 N(\{s_{ij}\})}{s_{13}s_{23}s_{14}s_{24}}$$

2. C_2^2:

$$\int \frac{d\Phi_3 N(\{s_{ij}\})}{(s_{34}s_{134})^2}, \int \frac{d\Phi_3 N(\{s_{ij}\})}{s_{34}^2 s_{134}s_{234}}$$

3. C_3^2:

$$\int \frac{d\Phi_3 N(\{s_{ij}\})}{(s_{13}s_{134})^2}, \int \frac{d\Phi_3 N(\{s_{ij}\})}{s_{13}s_{23}s_{134}s_{234}}$$

4. $C_1 \otimes C_2$:

$$\int \frac{d\Phi_3 N(\{s_{ij}\})}{s_{34}s_{234}s_{13}s_{24}}$$

5. $C_1 \otimes C_3$:

$$\int \frac{d\Phi_3 N(\{s_{ij}\})}{s_{134}s_{13}s_{23}s_{14}}, \int \frac{d\Phi_3 N(\{s_{ij}\})}{s_{134}s_{13}^2 s_{14}}$$

6. $C_2 \otimes C_3$:

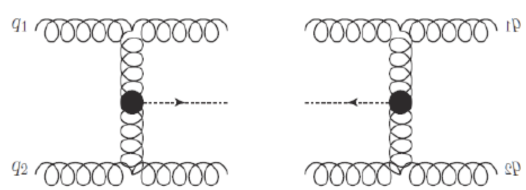
$$\int \frac{d\Phi_3 N(\{s_{ij}\})}{s_{34}s_{134}^2 s_{13}}, \int \frac{d\Phi_3 N(\{s_{ij}\})}{s_{34}s_{134}s_{234}s_{23}}$$

Original calculation by Anastasiou, Melnikov, Petriello **hep-ph/0501130**

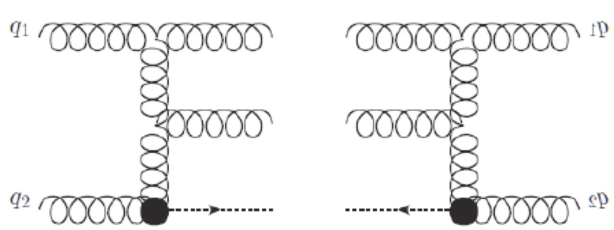
Applications in double real: Higgs revisited

The singularity structure of each individual integral depends strongly on the parametrization chosen.

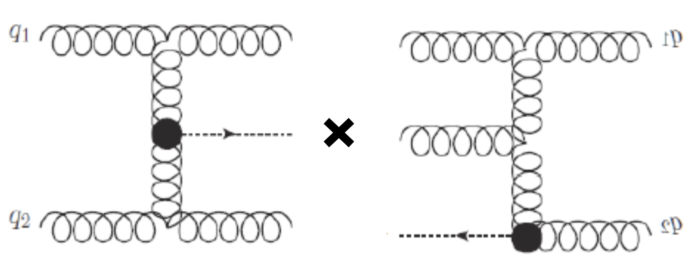
"Energy parametrization"



C_1^2 just partial fractioning factorizes it



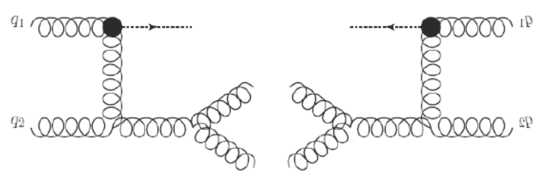
C_3^2 overlapping singularities that can be factored out with one or more non-linear mappings



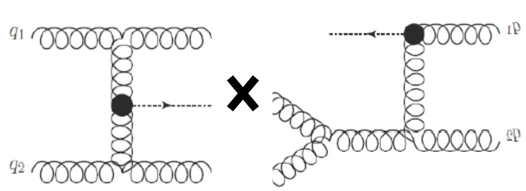
$C_1 \otimes C_3$ Line and overlapping singularities: the line singularity has to be treated first, along the lines of [Anastasiou, Melnikov, Petriello hep-ph/0501130] .

Applications in double real: Higgs revisited

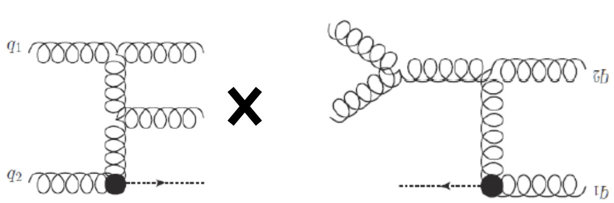
"Rapidity parametrization"



C_2^2 already in factorized form



$C_1 \otimes C_2$



$C_2 \otimes C_3$

Overlapping singularities that can be factored out with one or more non-linear mappings

Outlook

- A new approach to factorize overlapping singularities using non-linear mappings is pursued.
- As a result there is vast improvement with respect to previously used sector decomposition.
- This inspires great optimism for better NNLO calculations.
- Higgs production is revisited.
- More processes to come!